台大數學系 2021 特殊選才
面談筆試試題

1. 遞迴關係式:

(a) 費氏數列可以利用遞迴式 $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}, n \geq 3$ 來定義. 請解出 a_n 並求極限值

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}}.$$

(b) 數列 $\{a_n\}$ 是由遞迴式 $a_1 = 1, a_2 = 2, a_n = 2a_{n-1} + a_{n-2}, n \geq 3$ 來定義. 證明 $k \in \mathbb{N}$, $2^k \mid a_n$ 的充分必要條件是 $2^k \mid n$.

2. 考慮平面上兩條直線 L_1, L_2 以及一點 $P \notin L_1 \cup L_2$.

(a) 證明一定有 $Q \in L_1$, $R \in L_2$ 使得 PQR 是一個正三角形, 並討論有多少種可能. (如果你想用座標幾何, 可以假設 $P = (0,0)$, L_1 是 $y = 1$.)
(b) 如何用尺規作圖作出 Q, R?

3. 我們知道有理數在實數中有稠密性. 但其實這稠密性也有某種量化的版本.

(a) 證明: 存在某一個常數 $c > 0$ 使得對所有整數 p, q 其中 $q > 0$, 我們有

$$|q\sqrt{2} - p| > \frac{c}{q}.$$

(提示: c 的大小跟$\sqrt{2}$有關. 可以考慮先證明 $|q\sqrt{2} - p||q\sqrt{2} + p| \geq 1$.)

(b) 定義函數 $f(x)$ 在區間 $[1,2]$ 如下:

$$f(x) = \begin{cases} 0 & \text{如果 } x\text{爲無理數}, \\ \frac{1}{q^3} & \text{如果 } x = \frac{p}{q} \text{ 爲有理數並寫成最簡分數同時 } q > 0. \end{cases}$$

證明: $f(x)$ 在 $x = \sqrt{2}$ 可微, 同時 $f'(\sqrt{2}) = 0$. (你可以直接使用 (a).)

4. 令 $v_1 = (1,0,1,1), v_2 = (1,-1,1,1), v_3 = (1,1,-1,1), v_4 = (3,1,-1,3)$.

(a) v_1, v_2, v_3 是否線性獨立?
(b) v_1, v_2, v_3, v_4 是否線性獨立?
(c) 可否以 v_4 取代 v_1, v_2, v_3 其中之一, 而仍然展開相同線性空間?
(d) 可否以 v_4 取代 v_1, v_2, v_3 其中任一, 而仍然展開相同線性空間?

Date: **2021 年 12 月 19 日, 9:00-11:00. 請嚴謹作答, 缺乏實質內容或直接背誦公式並不會獲得任何分數. 入圍下午口試名單於 12:50 前公布.**

Here is the English translation: [1]

1. Recursive relations:

(a) The Fibonacci sequence is given by $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Solve a_n and find the limit $\lim_{n\to\infty} \frac{a_n}{a_{n-1}}$.

(2) Consider the sequence $\{a_n\}$: $a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$. Prove that $k \in \mathbb{N}$, $2^k \mid a_n$ if and only if $2^k \mid n$.

2. Consider two lines L_1, L_2 on a plane and a point $P \notin L_1 \cup L_2$.

(a) Show that there are points $Q \in L_1$, $R \in L_2$ such that PQR is an equilateral triangle. Discuss the number of possibilities for such (Q, R). (If you want to use coordinate geometry, you may assume that $P = (0, 0)$ and L_1 is $y = 1$.)

(b) How to construct Q and R by ruler and compass?

3. It is known that rational numbers are dense in real numbers. In fact, we also have a quantitative version of the denseness:

(a) Prove that there exists a constant $c > 0$ such that for all integers p, q with $q > 0$ we have

$$|q\sqrt{2} - p| > \frac{c}{q}.$$

(Hint: c depends on $\sqrt{2}$. You may try to prove $|q\sqrt{2} - p||q\sqrt{2} + p| \geq 1$ first.)

(b) Define $f(x)$ on $[1, 2]$ by the following:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{q^3} & \text{if } x = \frac{p}{q} \text{ is rational with the lowest form and } q > 0. \end{cases}$$

Prove that $f(x)$ is differentiable at $x = \sqrt{2}$ and $f'(\sqrt{2}) = 0$. (Hint: You can use the result of (a) directly.)

4. Let $v_1 = (1, 0, 1, 1)$, $v_2 = (1, -1, 1, 1)$, $v_3 = (1, 1, -1, 1)$, $v_4 = (3, 1, -1, 3)$.

(a) Determine wether v_1, v_2, v_3 are linearly independent?

(b) Determine wether v_1, v_2, v_3, v_4 are linearly independent?

(c) Can you substitute one of v_1, v_2, v_3 by v_4 so that they still form the same linear span?

(d) Can you substitute any one of v_1, v_2, v_3 by v_4 so that they still form the same linear span?

[1] Date: Am 9:00 - 11:00, December 19, 2021. Give your answers in details. No credit will be assigned to non-substantial solutions. The shortlist for the afternoon oral examination will be announced before 12:50