國立臺灣大學數學系

九十六學年度上學期博士班資格考試題

科目：統計

2007.09

1. Suppose the random variables $Y_1, \ldots, Y_n$ satisy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants and $\epsilon_1, \ldots, \epsilon_n$ are iid Normal$(0, \sigma^2)$, σ^2 unknown.

   (a) (10 pts.) Find the MLEs of β and σ^2.

   (b) (10 pts.) Show that $\sum Y_i / \sum x_i$ and the MLE in (a) are both unbiased estimators of β. Compare the variances of these two estimators.

2. (20 pts.) Find the LRT of

$$H_0 : \theta \le 0 \quad \text{versus} \quad H_1 : \theta > 0$$

based on a random sample $X_1, \cdots, X_n$ from a distribution with pdf

$$f(x|\theta, \lambda) = \frac{1}{\lambda} e^{-(x-\theta)/\lambda} I_{[\theta,\infty)}(x)$$

where both θ and λ are unknown, $-\infty < \theta < \infty$, $\lambda > 0$, and I is the indicator function.

3. Suppose X has the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1 + e^{(x-\theta)})^2}, \quad -\infty < x < \infty,$$

where $-\infty < \theta < \infty$ and θ is unknown.

   (a) (10 pts.) Show the the logistic location family has an MLR.

   (b) (10 pts.) Based on X, find the most powerful size α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Find the power of the test.

   (c) (5 pts.) Show that the test in (b) is UMP size α for testing $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$.

4. Let X be a single observation from the pdf $f(x) = \theta x^{\theta-1} I_{(0,1)}(x)$, where $\theta > 0$ is unknown and I is the indicator function.

   (a) (10 pts.) Let $T = X^\theta$. Show that T is a pivotal quantity. Use T to construct $1 - \alpha$ confidence intervals for θ.

   (b) (5 pts.) Among the confidence intervals in (a), show that $\{\theta : 0 \le \theta \le \log \alpha / \log X\}$ has the minimum length.

5. Let $X_1, \ldots, X_n$ be iid random variables from a Poisson(λ) distribution with the pmf

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots.$$

   (a) (10 pts.) Find the best unbiased estimator of $\lambda e^{-\lambda} = P(X_1 = 1)$.

   (b) (10 pts.) Find a UMA one-sided $1 - \alpha$ confidence interval for λ of the form $(0, U(X)]$.