國立臺灣大學數學系
九十五學年度博士班資格考試試題
科目：統計

2007.06.01

1. (20 pts) Let $X_1, \ldots, X_n$ be i.i.d. having the Poisson distribution $Poisson(\theta)$ with $\theta > 0$.

   (a) (12 pts) Find the UMVUE of $\exp(-t\theta)$ with a fixed $t > 0$.

   (b) (8 pts) Show that the ratio of asymptotic mean squared error of UMVUE T_n and Cramer-Rao lower bound goes to 1.
   (You can either show it or cite a theorem but the conditions should be checked.)

2. (20 pts) Let $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ be i.i.d. with the p.d.f.

$$\lambda^{-1}\mu^{-1}e^{-y/\lambda}e^{-z/\mu}I_{(0,\infty)}(y)I_{(0,\infty)}(z),$$

   where $\lambda > 0$ and $\mu > 0$. Here I_A is the indicator function of A.

   (a) (5 pts) Find the MLE of (λ, μ).

   (b) (15 pts) Suppose that we only observe $X_i = \min(Y_i, Z_i)$ and $\triangle_i = 1$ if $X_i = Y_i$ and $\triangle_i = 0$ if $X_i = Z_i$. Determine the MLE of (λ, μ).

3. (20 pts) Assume that

$$X_{i+1} - \theta = \rho(X_i - \theta) + \delta_i,$$

   with δ_i i.i.d. $N(0,1)$ and $X_1 \sim N(\theta, 1/(1-\rho^2))$.

   (a) (10 pts) Determine $Corr(X_1, X_i)$ and $Var(X_i)$.

   (b) (10 pts) Show that $\bar{X}$ is a consistent estimator of θ.

4. (20 pts) Let $X_1, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$. Suppose that $\sigma^2 = \gamma\mu^2$ with unknown $\gamma > 0$ and $\mu \in R$. Obtain a confidence set for γ with confidence coefficient $1 - \alpha$ by inverting the acceptance region of the likelihood ratio tests for

$$H_0 : \gamma = \gamma_0 \text{ versus } H_1 : \gamma \neq \gamma_0.$$

5. (20 pts) Assume that

$$X_i = \theta t_i + \epsilon_i, \quad i = 1, \ldots, n,$$

   where $\theta \in \Theta$ is an unknown parameter, Θ is a closed subset of R, ϵ's are i.i.d. uniform random variables on the interval $[-1, 1]$, t_i's are fixed constants, $\sup_i |t_i| < \infty$ and $\sup_i t_i - \inf_i t_i > 2$. Let

$$T_n = S_n(\tilde{\theta}_n) = \min_{\gamma \in \Theta} S_n(\gamma),$$

   where

$$S_n(\gamma) = 2 \max_{i \leq n} |X_i - \gamma t_i| / \sqrt{1 + r^2}.$$

   (a) (8 pts) Show that the sequence $\{\tilde{\theta}_n, n = 1, 2, \ldots\}$ is bounded.

   (b) (12 pts) Show that T_n is a strongly consistent estimator of

$$\vartheta = \min_{\gamma \in \Theta} S(\gamma),$$

   where $S(\gamma) = \lim_{n \to \infty} S_n(\gamma)$ almost surely.