臺灣大學數學系

103 學年度下學期博士班資格考試題

科目：統計 2015.03.06

1. Let $X \sim Hypergeometric(N, M, K)$, $Y \sim NegativeBinomial(r, p)$, $Z \sim Poisson(\lambda)$, and $W \sim \chi^2_p$.

(1a) (5%) State and show the conditions so that the probability distribution of X can be approximated via the probability distribution of Z.

(1b) (5%) State and show the conditions so that the probability distribution of Y can be approximated via the probability distribution of Z.

(1c) (5%) Express the cumulative distribution function of Z via the cumulative distribution function of W.

2. (15%) Let $X_1, \ldots, X_n$ be a random sample from the probability density function $f_X(x) = (ax^{a-1}/\theta^a) I_{\{(0,\theta)\}}(x)$ and $X_{(1)} < \ldots < X_{(n)}$ stand for the corresponding order statistics. Find the joint distribution of $(X_{(1)}/X_{(2)}, X_{(2)}\ /X_{(3)}, \ldots, X_{(n-1)}/X_{(n)})^\top$.

3. (5%) (5%) Give examples to show that convergence in probability cannot imply the convergence almost surely and convergence in L^p.

4. (10%) Show that $\lim_{n\to\infty} \sum_{k=0}^{n} e^{-n}n^k/k! = 0.5$ and $\lim_{n\to\infty} \int_0^n u^{n-1}e^{-u}du/\Gamma(n) = 0.5$.

5. (15%) (5%) Let $X_1, \ldots, X_n$ be a random sample from a population with a probability density function

$$f_X(x|\theta) = \theta x^{\theta-1},\ \ 0 < x < 1,\ \ 0 < \theta < \infty.$$

Find the uniformly minimum variance unbiased estimator (UMVUE) of θ and show that the variance of the UMVUE cannot attain the Cramér-Rao lower bound.

6. Let $X_1, \cdots, X_n$ be a random sample from a population with probability density function

$$f(x|\theta, \nu) = \frac{\theta\nu^\theta}{x^{\theta+1}} 1_{[\nu,\infty)}(x),$$

where θ and ν are unknown positive parameters.

(6a) (8%) Find the maximum likelihood estimators of θ and ν.

(6b) (7%) Show that the likelihood ratio test of $H_0 : \theta = 1$ versus $H_A : \theta \neq 1$ has critical region of the form $\{(X_1, \ldots, X_n) : T(X_1, \cdots, X_n) \leq c_1 \ or\ T(X_1, \ldots, X_n) \geq c_2\}$, where $0 < c_1 < c_2$ and $T(X_1, \ldots, X_n) = \ln\left(\frac{\prod_{i=1}^n X_i}{(min\{X_1, \ldots, X_n\})^n}\right)$.

7. (15%) Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with rate λ. Find a uniformly most accurate $(1-\alpha)$ confidence interval for λ.