# 臺灣大學數學系
## 100 學年度上學期博士班資格考試題
## 科目：統計

2011.09.15

1. (20%) Suppose $X_1, \dots, X_n$ are iid $N(\mu, \mu^2)$, $\mu > 0$.

   (a) (5%) Show that both $\bar{X}$ and $s$ are both consistent estimates of $\mu$. Here $s^2$ is the sample variance.

   (b) (8%) Derive joint limiting distribution of $(\bar{X}, s^2)$.

   (c) (7%) Find the limit of $P\left(|s-\mu| < |\bar{X}-\mu|\right)$.

2. (15%) Let the distribution of survival times of patients receiving a standard treatment be the known distribution $F_0$, and let $Y_1, \dots, Y_n$ be the i.i.d. survival times of a sample of patients receiving an experimental treatment. Assume that the distribution of survival times of $Y$ is of the form $G(y, \Delta)$ where $G(y,\Delta) = 1 - [1 - F_0(y)]^{\Delta}$ for $y > 0$ and $\Delta > 0$. To test whether the new treatment is beneficial we test $H_0 : \Delta \le 1$ versus $H_A : \Delta > 1$. Assume that $F_0$ has a density $f_0$.

   (a) (8%) Find the UMP test.

   (b) (7%) Show how to find critical values.

3. (20%) Suppose $X_1, X_2, \dots, X_n \overset{iid}{\sim} U[\mu - \sigma, \mu + \sigma]$, and we want to test $H_0 : \mu = 0$ versus $H_A : \mu \neq 0$ based on $X_1, X_2, \dots, X_n$.

   (a) (6%) Derive the likelihood ratio test and express it as a function of $W = X_{(n)} - X_{(1)}$ and $U = \max(X_{(n)}, -X_{(1)})$.

   (b) (7%) Show that, under $H_0$, $U$ is complete and sufficient and $U$ and $W/U$ are independent.

   (c) (7%) Derive the asymptotic distribution of $-2\log\Lambda_n$ where $\Lambda$ is the likelihood ratio.

4. (15%) Suppose $X_1, X_2, \dots$ are iid $Poisson(\mu)$ and we wish to estimate $P(X_1 = 0)$.

   (a) (4%) Derive the maximum likelihood estimate of $P(X_1 = 0)$ which is denoted by $\delta_n$.

   (b) (5%) Derive the approximations of $E(\delta_n) - P(X_1 = 0)$ and $Var(\delta_n)$ with an error bound $O(n^{-2})$. Please justify your answer.

   (c) (6%) Show that $(1 - 1/n)^{\sum_{i=1}^n X_i}$ is UMVUE of $P(X_1 = 0)$.

5. (15%) Let $X_1, X_2, \dots, X_n$ be the indicators of $n$ Bernoulli trials with success probability $\theta$. Suppose $\ell(\theta, a)$ is the quadratic loss $(\theta - a)^2$ and that the prior $\pi(\theta)$ is the beta, $\beta(r,s)$, density. (i.e., The beta density function is $x^{r-1}(1-x)^{s-1}/B(r,s)$.)

   (a) (7%) Find the Bayes estimate $\hat{\theta}_B$ of $\theta$.

   (b) (8%) Show that $\hat{\theta} = (S_n + \sqrt{n}/2)/(n + \sqrt{n})$ has a constant risk and $\hat{\theta}$ is minimax. Here $S_n = \sum_{i=1}^n X_i$.

6. (15%) Let $P_\theta$ denote the uniform distribution on $[0, \theta]^2$, for $\theta > 0$. That is, the coordinates $X_1$ and $X_2$ are independent $Uniform[0, \theta]$ under $P_\theta$. Let $S = X_1 + X_2$ and $M = \max(X_1, X_2)$. Consider estimation of $\theta$ with loss function $L(\theta, a) = (\theta - a)^2$.

(a) (5%) Explain why $E_\theta(S|M = m)$ is preferred to $S$ for estimating $\theta$.

(b) (5%) Explain why $E_\theta(2X_1|S)$ is preferred to $2X_1$ for estimating $\theta$.

(c) (5%) Explain why $E_\theta(3M/2|S = s)$ is not preferred to $3M/2$ for estimating $\theta$.