臺灣大學數學系
99 學年度下學期博士班資格考試題

科目：實分析 2011.02.24

1. (10 points)
Prove the following integral version of Minkowski's inequality for $1 \le p < \infty$:

$$[\int \left| \int f(x,y)dx \right|^p dy]^{1/p} \le \int [\int \left| f(x,y) \right|^p dy]^{1/p} dx.$$

2. (10 points)
Prove that $L^\infty(E)$ is not separable for any measurable set E with positive measure.

3. (10 points)
Prove that a continuous function defined on a closed interval is uniformly continuous.

4. (10 points)
Prove that a function defined on a closed interval is Riemann integrable.

5. (30 points)

(i) State and prove the simple Vitali covering lemma.

(ii) Prove that if $f \in L(\mathbb{R}^n)$ and f has compact support, then the set

$$A_{f^*}(\alpha) = \{x \in \mathbb{R}^n | f^*(x) > \alpha\}$$

is bounded in $\mathbb{R}^n$, where f^* is the Hardy-Littlewood maximum function of f.

(iii) Under the assumption of (ii), show that there exists a positive constant c independent of f and α such that $\omega_{f^*}(\alpha) = |A_{f^*}(\alpha)| \le \frac{c}{\alpha} \int_{\mathbb{R}^n} |f|$

6. True or False. For each of the following statements, prove it if the statement is true; otherwise, give a counter example. (30 points)

a) Suppose $f(x) = \sum_{j=0}^{\infty} a_j x^j$, where $a_j \in \mathbb{R}^1$, $j = 0, 1, 2, \cdots$, and $\sum_{j=0}^{\infty} |a_j| = 10252010$. Then $f(x)$ is of bounded variation on the closed interval $[0,1]$.

b) Suppose that $g(x)$ is a bounded function on $[0,1]$ such that for any positive $\epsilon < 1$, $V[g;\epsilon,1] < M$. Then $V[g;0,1] \le M$, where $V[g;a,b]$ is the variation of g over $[a,b]$.

c) Suppose that $f(x)$ and $\phi(x)$ are bounded functions on $[-1,1]$ such that $\int_{-1}^{0} f(x)d\phi(x) = 1$ and $\int_0^1 f(x)d\phi(x) = 1$. Then $\int_{-1}^{1} f(x)d\phi(x) = 2$.

d) $f_n(x) \to f(x)$ uniformly in $\mathbb{R} \Rightarrow f_n(x) \to f(x)$ in $L^1(\mathbb{R})$.

e) $f_n \to f$ in $L^1([0,1]) \Rightarrow f_n \to f$ pointwise a.e on $[0,1]$.