臺灣大學數學系
99 學年度上學期博士班資格考試題
科目：實分析 2010.09.17

1. (20 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Must the following statement hold?
Given $\delta > 0$, there exists a continuous function ϕ on $\mathbb{R}$ such that

$$m\left(\{x \in \mathbb{R} : f(x) \neq \phi(x)\}\right) < \delta .$$

Hereafter, $m(\cdot)$ is the Lebesgue measure. Prove or disprove your answer.

2. (20 pts) Let $f \in L^2(\mathbb{R}^n)$ and $K(x) = \frac{1}{1+|x|}$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, where $|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$. Set $K_\epsilon(x) = \frac{\epsilon^{1-n}}{\epsilon+|x|}$ and let $\tilde{f}$ be the maximal function defined by

$$\tilde{f}(x) = \sup_Q \frac{1}{m(Q)} \int_Q |f(y)|\, dy ,$$

where the supremum is taken over all cubes Q with center x and edges parallel to the coordinate axes.

(i) Must $\tilde{f} \in L^2(\mathbb{R}^n)$? (10 pts)

(ii) Must there exist a constant C independent of f such that

$$\sup_{\epsilon>0} |K_\epsilon * f(x)| \leq C\, \tilde{f}(x) , \quad \forall x \in \mathbb{R}^n$$

holds? Here $*$ is the convolution. (10 pts)

Prove or disprove all your answers.

3. (20 pts) Let $\{f_k : k = 1, 2, 3, \cdots\}$ be a bounded sequence in $L^2([0,1])$. Must the sequence $\{f_k\}$ have a convergent subsequence in $L^1([0,1])$? Prove or disprove all your answers. (10 pts)

4. (20 pts) Can there exist $f \in L^1(\mathbb{R})$ and a sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R})$ such that as $n \to \infty$, $\|f_n - f\|_{L^1} \to 0$ but $f_n(x) \to f(x)$ for no x? Prove or disprove your answer.

5. (20 pt) Let $E \subset [0, 2\pi]$ be a measurable set with positive (Lebesgue) measure. Must $\lim_{n\to\infty} \int_E \cos^2(nx + u_n)\, dx = 0$ for any sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}$? Prove or disprove your answer.