臺灣大學數學系
98 學年度下學期博士班資格考試題
科目：實分析

2010.02.25

Please answer all questions as complete as possible. You will NOT get partial credits for incomplete solutions. Passing grade: 70.

1.(20%) Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that if $\mu(X) < \infty$, $f_n \to f$ in measure and $g_n \to g$ in measure, then $f_n g_n \to fg$ in measure. For this statement, can the assumption of $\mu(X) < \infty$ be removed?

2. True or false:
(a)(10%) Let f be a function on $\mathbb{R}^n$. The set of discontinuities of f is Borel set.
(b)(10%) $|f|$ is measurable $\Rightarrow$ f is measurable.
(c)(20%) Let f be a nonnegative absolutely continuous function on $[a, b]$, then f^s is absolutely continuous for any $0 < s < 1$.

3. Let $\mathcal{Z}$ be the space of measurable, finite-valued a.e. functions on the finite measure space $(X, \mathcal{B}, \mu)$. For $f, g \in \mathcal{Z}$, define

$$d(f,g) = \int \frac{|f-g|}{1+|f-g|} d\mu.$$

Show that
(a)(10%) for $f_n \in \mathcal{Z}$, $f_n \to 0$ in measure if and only if $d(f_n, 0) \to 0$.
(b)(20%) $\mathcal{Z}$ is a complete metric space with respect to d.

4.(20%) Let $(X, \mathcal{B}, \mu)$ be a σ finite measure space and $(\mathbb{R}, \mathcal{R}, \lambda)$ be the Lebesgue measure space. Assume that $f : X \to [0, \infty)$ is measurable. Show that the set $\Gamma_f = \{(x,t) \in X \times \mathbb{R} : f(x) = t\}$ is an element of $\mathcal{B} \times \mathcal{R}$ (product σ field) and $\mu \times \lambda(\Gamma) = 0$.

5.(10%) Let S be the set of positive a.e. functions on $[a, b]$. Define

$$M_f = \left(\int_a^b f dx\right)\left(\int_a^b f^{-1} dx\right).$$

Determine $\min_{f \in S} M_f$ and classify all minimizers of $\min_{f \in S} M_f$.