臺灣大學數學系
九十八學年度上學期博士班資格考試題
科目：實分析 2009.09.17

1. (30 pts) Let f be measurable and the derivative $f' = 0$ almost everywhere on $\mathbb{R}$.

(1) Must f be a constant function? (10 pts)

(2) Suppose that f is absolutely continuous on $\mathbb{R}$. Must f be a constant function? (10 pts)

(3) Must f be Riemann integrable? (10 pts)

Prove or disprove all your answers.

2. (20 pts) Let f be measurable and periodic with period 1 i.e. $f(t+1) = f(t)$ for $t \in \mathbb{R}$. Suppose that there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)|\, dt \leq c,$$

for all a and b. Here c is independent of a and b. Must f be Lebesgue integrable on $(0,1)$? Prove or disprove your answer.

3. (30 pts) Let f be an analytic function on Ω, where Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$. Let $P(x_1, x_2) = \sum_{i,j=0}^{10} a_{i,j} x_1^i x_2^j$ for $x_1, x_2 \in \mathbb{R}$, where $a_{i,j}$'s are constants. Suppose that the set $\{(x_1, x_2) \in \mathbb{R}^2 : P(x_1, x_2) = 0\}$ has positive Lebesgue measure in $\mathbb{R}^2$. Must P be the zero function (10 pts)? Suppose that the set $\{x \in \Omega : f(x) = 0\}$ has positive Lebesgue measure in $\mathbb{R}^n$. Must f be the zero function (20 pts)? Prove or disprove your answer. Note: A function is analytic if and only if it is equal to its Taylor series in some neighborhood of every point.

4. (20 pts) Let $\phi \in L^\infty(B_1)$ and $\phi \geq 0$ almost everywhere, where $B_1 = \{x \in \mathbb{R}^n : \|x\| < 1\}$ is the unit ball of $\mathbb{R}^n$ with the center at the origin. Must the following inequality

$$\frac{\int_{B_1} \phi\, e^\phi}{\int_{B_1} e^\phi} \geq \frac{\int_{B_1} \phi\, e^{-\phi}}{\int_{B_1} e^{-\phi}}$$

hold? Prove or disprove your answer.