臺灣大學數學系
九十七學年度上學期博士班資格考試題
科目：實分析 2008.09.18

Real Analysis Qualifying Exam

There are six problems in this exam. To pass the exam, you have to answer at least four problems correctly.

1. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. Assume that $f : X \to [0, \infty)$ is a measurable function. Let $\varphi : [0, \infty) \to [0, \infty)$ be a C^1 function with never-vanishing derivative. Show that

$$\int_X \varphi \circ f(x) d\mu = \int_0^\infty \varphi'(t)\mu(\{x : f(x) > t\})dt.$$

2. Prove or disprove the following statements.
(a) Denote $\mathcal{L}$ the Lebesgue measurable set of $\mathbb{R}$. Assume that $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable functions. i.e., $\{x : f(x) > t\} \in \mathcal{L}$ and $\{x : g(x) > t\} \in \mathcal{L}$ for all $t \in \mathbb{R}$. Is $f \circ g$ Lebesgue measurable?
(b) Let $\mathcal{B}$ be the Borel set of $\mathbb{R}$. Assume that f and g are Borel measurable, i.e., $\{x : f(x) > t\} \in \mathcal{B}$ and $\{x : g(x) > t\} \in \mathcal{B}$ for all $t \in \mathbb{R}$. Is $f \circ g$ Borel measurable?
(c) Let μ be the Lebesgue measure of $\mathbb{R}$. Let A be a Lebesgue measurable set and B be a Borel set satisfying $\mu(A) = \mu(B) = 0$. If $N \subset A$, then must N be a Lebesgue measurable set? On the other hand, if $N \subset B$, then must N be a Borel set?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that there exists a Borel function g such that $f = g$ a.e.

4. Prove or disprove the following statements (consider $\mathbb{R}$ with Lebesgue measure).
(a) $f_n \to f$ uniformly in $\mathbb{R} \Rightarrow f_n \to f$ in L^1.
(b) $f_n \to f$ in $L^1 \Rightarrow f_n \to f$ pointwise a.e.
(c) Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ a.e., then $\psi \circ f_n \to \psi \circ f$ a.e.

5. Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < \infty$. Assume that the real-valued function f is measurable and $\lim \int f^n d\mu$ exists and is finite. Show that

$$\lim \int f^n d\mu = \mu(\{x : f(x) = 1\}).$$

6. Let $f : X \to [0, \infty)$ be a measurable and essentially bounded function. Denote ess sup $f =: M > 0$. Assume that $\mu(X) < \infty$. Show that
(a)

$$\lim \Big(\int f^n d\mu \Big)^{1/n} = M.$$

(b)

$$\lim \frac{\int f^{n+1} d\mu}{\int f^n d\mu} = M.$$