國立臺灣大學數學系
九十六學年度上學期博士班資格考試題
科目：實分析

2007.09

There are six problems in this exam. To pass the exam, you have to at least answer four problems correctly.

1. If f is measurable on E, define $\omega_f(a) = m\{f > a\}$ for $-\infty < a < +\infty$. Hereafter $m(A)$ denotes the measure of A. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \to f$ *in measure*, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. [For the second part, show that if $f_k \to f$ in measure, then $\limsup_{k\to\infty} \omega_{f_k}(a) \le \omega_f(a-\varepsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \ge \omega_f(a+\varepsilon)$ for every $\varepsilon > 0$.]

2. If $p > 0$, $\int_E |f - f_k|^p \to 0$, and $\int_E |f_k|^p \le M$ for all k, show that $\int_E |f|^p \le M$.

3. Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

4. Prove the following *integral version of Minkowski's inequality* for $1 \le p < \infty$:

$$\Big[\int \Big|\int f(x,y)dx\Big|^p dy\Big]^{1/p} \le \int \Big[\int |f(x,y)|^p dy\Big]^{1/p} dx.$$

[For $1 < p < \infty$, note that the pth power of the left-hand side is majorized by $\int\int [\int |f(z,y)| dz]^{p-1} |f(x,y)| dx dy$. Integrate first with respect to y and apply Hölder inequality.]

5. A sequence $\{f_n\}$ of measurable functions is said to be a Cauchy sequence in measure if given $\epsilon > 0$ there is an N such that for all $m, n \ge N$ we have

$$m\{x : |f_n(x) - f_m(x)| \ge \epsilon\} < \epsilon.$$

Show that if $\{f_n\}$ is Cauchy sequence in measure, then there is a function f to which the sequence $\{f_n\}$ converges in measure. [Choose $n_{\nu+1} > n_\nu$ so that $m\{x : |f_{n_\nu} - f_{n_{\nu+1}}| > 2^{-\nu}\} < 2^{-\nu}$. Then the series $\sum (f_{n_{\nu+1}} - f_{n_\nu})$ converges almost everywhere to a function g. Let $f = g + f_{n_1}$. Then $f_{n_\nu} \to f$ in measure, and one can show that consequently $f_n \to f$ in measure.]

6. Let $\{f_n\}$ be a sequence of functions in L^p, $1 \le p < \infty$, which converges almost everywhere to a function f in L^p. Show that $\{f_n\}$ converges to f in L^p if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$.