臺灣大學數學系111學年度第2學期博士班一般資格考試

科目：實分析

2023.02.24

1. (10%) Let $A \subset \mathbb{R}$ has Lebesgue measure zero and let
$$f(x) = \begin{cases} \dfrac{1}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Show that $B = \{f(x) : x \in A\}$ has Lebesgue measure zero.

2. (20%) Show that a subset Y of $\mathbb{R}^n$ is Lebesgue measurable if and only if for every set $A \subset \mathbb{R}^n$,
$$|A|_e = |A \cap Y|_e + |A \setminus Y|_e,$$
where $|A|_e$ denotes the Lebesgue outer (exterior) measure of A.

3. (1) (25%) Let f_k be a Cauchy sequence in $L^1(\mathbb{R}^n)$. Show that there exist a function $f \in L^1(\mathbb{R}^n)$ and a subsequence $f_{k_j}, j = 1, 2, 3, ...$, such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ as $k \to \infty$ and $f_{k_j} \to f$ a.e. as $j \to \infty$.
(2) (15%) Let $A_k \subset [0,1], k = 1, 2, 3, ...$, be a sequence of Lebesgue measurable sets with
$$\sum_{k=1}^{\infty} |A_k| < \infty,$$
where $|A_k|$ denotes the Lebesgue measure of A_k. Let χ_{A_k} denote the characteristic function of A_k. Show that $\chi_{A_k} \to 0$ in $L^1([0,1])$ and $\chi_{A_k} \to 0$ a.e. on $[0,1]$ as $k \to \infty$.

4. (1) (10%) Prove Young's inequality.
(2) (20%) Prove Hölder's inequality for functions defined on $\mathbb{R}^n$.