臺灣大學數學系111學年度第1學期博士班一般資格考試

科目：實分析

2022.09.16

1. Let f be a real-valued function on $\mathbb{R}$.

(a) (10%) If f is Lebesgue measurable, then for any $a \in \mathbb{R}$, $f^{-1}(a)$ is a measurable set.

(b) (15%) Does the converse of (a) hold true? That is, if for any $a \in \mathbb{R}$, $f^{-1}(a)$ is measurable, then f is a measurable function.

2. Let $1 \le p, q \le \infty$ be conjugate, i.e., $1/p + 1/q = 1$. Let $\{\mathbf{x}_n = (x_{j,n})_{j=1}^{\infty}\}$ be a sequence of elements in ℓ_p. Recall that the sequence $\{\mathbf{x}_n\}$ converges weakly in ℓ_p to some $\mathbf{x} = (x_j) \in \ell_p$ if

$$\lim_{n\to\infty} (\mathbf{x}_n, \mathbf{y}) = \lim_{n\to\infty} \sum_j x_{j,n} y_j = \sum_j x_j y_j, \ \ \forall \ \mathbf{y} = (y_j) \in \ell_q.$$

Show that

(a) (5%) strong convergence implies weak convergence;

(b) (15%) for $1 < p \le \infty$, the converse is false by constructing a counterexample;

(c) (15%) for $p = 1$, weak and strong convergence are equivalent.

3.(25%) Let f be a measurable function on $\mathbb{R}^n$. Define $\langle f \rangle = \sup_{\alpha>0} \alpha |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|$, and recall that f belongs to weak $L^1(\mathbb{R}^n)$ iff $\langle f \rangle < \infty$. Show that weak $L^1(\mathbb{R}^n)$ has all the properties of a Banach space with respect to $\langle \cdot \rangle$ except the triangle inequality. Show however that there is a constant $\kappa > 1$ such that the quasi-triangle inequality $\langle f + g \rangle \le \kappa(\langle f \rangle + \langle g \rangle)$ holds for all measurable functions f and g. Also show that the constant κ cannot be 1 by considering the case of one dimension $f = \chi_{[0,1/2]} + 2\chi_{(1/2,1]}$, $g = 2\chi_{[0,1/2]} + \chi_{(1/2,1]}$.

4.(15%) Let f be a real-valued function on $\mathbb{R}$. Then the set of discontinuous points of f is an F_σ set.