臺灣大學數學系110學年度第2學期博士班一般資格考試

科目：實分析 2022.04.14

Determine whether the following statements are true or false. Give a proof or a counter example for your answer.

1. Let $k(x,y)$ be an measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that both

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \int |k(x,y)|dx \text{ and } \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int |k(x,y)|dy$$

are finite. Then, for $1 \le p < \infty$, there exists $C > 0$ such that

$$\int \left| \int k(x,y)f(y)dy \right|^p dx \le C \int |f(x)|^p dx.$$

2. Let $\{f_k\}$ be an sequence of measurable functions increasing to f at any point $x \in \mathbb{R}^n$. Suppose $\int f_k(x)dx$ exists for every $k \in \mathbb{N}$. Then,

$$\lim_{k\to\infty} \int f_k(x)dx = \int f(x)dx.$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose f^2 is a measurable function. Then, f is a measurable function.

4. f^* denotes the Hardy-Littlewood maximal function of f. Assume $f \in L(\mathbb{R}^n)$. $f^* \in L(\mathbb{R}^n)$ if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

5. Let f be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y)dy \right) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y,x)dx \right) dy = a,$$

where a is a finite real number. Then, f is integrable. Furthermore,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,y)dxdy = a.$$

6. Let $\{f_k\}$ be a sequence in $L(\mathbb{R}^n)$. Such that

$$\int |f_k - f_j|dx \to 0 \text{ as } k, j \to \infty.$$

Then, there exists $f \in L(\mathbb{R}^n)$ such that $\{f_k\}$ converges to f as $k \to \infty$. at almost every point in $\mathbb{R}^n$.

7. Suppose that $\{f_k\}$ is a sequence of measurable functions which converges everywhere in a set E of finite measure to a finite limit f. Then, given $\epsilon > 0$, there is a closed subset F of E such that $|E \setminus F| < \epsilon$ and

$$\lim_{k\to\infty} \int_F f_k dx = \int_F f dx.$$

Here, $E \setminus F = \{x | x \in E,\ x \notin F\}$.