臺灣大學數學系

110 學年度上學期博士班資格考試題

科目：實分析 2021.09.24

1.(15%) Let B be a Lebesgue nonmeasurable set in $\mathbb{R}^n$. Prove that there exists a set $B_0 \subset B$ such that B_0 is also nonmeasurable and for any $A \subset B_0$ if A is measurable, then $|A| = 0$.

2.(10%) Suppose f_k converges a.e to f and $||f_k||_p \to ||f||_p$ for some $0 < p < \infty$. Show that $||f_k - f||_p \to 0$.

3.(15%) Assuming f is nonnegative and integrable in $\mathbb{R}^1$. If f is also Lipschitz, i.e $|f(x) - f(y)| \leq C|x - y|$. Prove that $\liminf_{n\to\infty} \sqrt{n} f(n) = 0$.

4.(15%) Let A be a Lebesgue measurable set in $\mathbb{R}^1$. Prove that for any $\epsilon > 0$, there exists a continuous function $f(x)$ in $\mathbb{R}^1$ such that $|\{x \in \mathbb{R}^1; f(x) \neq \chi_A(x)\}| < \epsilon$.

5.(15%) Show that $\liminf_{p\to\infty} ||f||_{L^p(E)} \geq ||f||_{L^\infty(E)}$ for any measurable function f and measurable set E.

6.(15%) Given a measurable function $f(x)$ in $\mathbb{R}^n$ and suppose that $|\{x, |f(x)| > \lambda\}| \leq \frac{A_1}{\lambda}$ and $|\{x, |f(x)| > \lambda\}| \leq \frac{A_2}{\lambda^r}$ for some $1 < r$. Show that $f \in L^p(\mathbb{R}^n)$ for any $1 < p < r$.

7.(15%) Given $f \in L^p(\mathbb{R}^1)$ for some $1 \leq p < \infty$. Given $t \in \mathbb{R}^1$, define $f_t(x) := f(x - t)$. Show that $\lim_{t\to 0} ||f_t - f||_p = 0$.