PH. D QUALIFYING EXAM : REAL ANALYSIS 2021 SPRING

1. (30 %)

  (i) Show that $\exp(x)$ is a convex function.

  (ii) State and prove the Young's inequality (you may use the result from (i)).

  (iii) State and prove the Hölder's inequality. (for vectors in $\mathbb{R}^N$)

  (iv) Let $1 < p < \infty$. Show that for any $u, v \in \mathbb{R}^N$,

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

2. (20 %) Assume $f(x)$ is an odd real continuous function defined on $\mathbb{R}$, find the value of the limit

$$\lim_{n \to \infty} n \int_0^{\frac{2}{n}} f(x + \frac{1}{n}) \sin(nx) dx.$$

3. (25 %) Calculate

$$\int_0^\infty \frac{\sin x}{x e^x} dx.$$

4. (25 %)Prove the following integral version of Minkowski's inequality for $1 \leq p < \infty$

$$[\int |\int f(x,y) dx|^p dy]^{\frac{1}{p}} \leq \int [\int |f(x,y)|^p dy]^{\frac{1}{p}} dx.$$