臺灣大學數學系
109 學年度上學期博士班資格考試題
科目：實分析　　2020.09.17

PH.D QUALIFYING EXAM : REAL ANALYSIS 2020 FALL

1.(10%) Suppose a function $f(x)$ on $[0,1]$ is defined as the following. If $x \in [0,1]$ is a rational number, i.e. $x = \frac{q}{p}$ for some integers p and q, then define $f(x) = \frac{1}{p}$, when x is not rational, define $f(x) = 0$. Prove or disprove f is measurable.

2.(15%) Let Q be the unit square in $\mathbb{R}^2$. Suppose $\{f_n\}$ is a sequence of non-negative measurable functions in $L^1(Q)$ and $\lim_{n\to\infty} \int_Q f_n = \int_Q f < \infty$ and $f_n(x) \to f(x)$ pointwise. Show that $\lim_{n\to\infty} \int_E f_n = \int_E f$ for any measurable set $E \subset Q$.

3.(15%) Let $f(x) \in L^3(\mathbb{R})$ and $\phi(x) = \sin(\pi x) \cdot \chi_{[-1,1]}(x)$. Define

$$f_n(x) = n \int f(x-y)\phi(ny)dy.$$

Show that f_n converges to 0 a.e.

4.(15%) Suppose μ is a Borel measure on $\mathbb{R}^n$ and assume that there exists a constant $c > 0$ such that whenever a Borel set E satisfies $|E| = c$, then $\mu(E) = c$. Show that μ is absolutely continuous w.r.t Lebesgue measure.

5.(15%) Let $1 < p < r < q < \infty$ and define θ by $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. Show that $||f||_r \le ||f||_p^\theta ||f||_q^{1-\theta}$.

6.(15%) Denote B the Borel σ-algebra in $[0,1]$, and $M([0,1])$ be the space of real finite measures $\mu : B \to \mathbb{R}$ with norm $||\mu|| = |\mu|([0,1])$. Show that $M([0,1])$ is a real Banach space.

7.(15%) (a): Suppose E_1 and E_2 are nonmeasurable sets in $\mathbb{R}^n$. Prove or disprove $E_1 \cup E_2$ is nonmeasurable.
(b): Prove or disprove the following statement. B is not Lebesgue measurable if and only if there exists $\epsilon > 0$ such that for every Lebesgue measurable set $A \subset B$, $|B - A|_e \ge \epsilon$.