臺灣大學數學系
108 學年度上學期博士班資格考試題
科目：實分析 2019.09.11

PH.D QUALIFYING EXAM : REAL ANALYSIS 2019 FALL

1.(10%) Denote Mf the Hardy-Littlewood maximal function of $f \in L^1(\mathbb{R}^n)$. Show that Mf is of weak $(1,1)$.

2.(15%) Assume $f(x)$ is a real valued continuous function on $\mathbb{R}$ and $\int_{\mathbb{R}} |f(x)|dx < \infty$. Show that there exists a sequence $\{x_n\}$ of real numbers such that $x_n \to \infty$, $x_n f(x_n) \to 0$ and $x_n f(-x_n) \to 0$.

3.(15%) Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} \geq 1$. Assume r satisfies $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, show that $||f * g||_r \leq ||f||_p ||g||_q$.

4.(15%) Assume $f, g \in L^1(\mathbb{R}^n)$. Given a real value λ, let $F_\lambda = \{x \in \mathbb{R}^n; f(x) > \lambda\}$ and $G_\lambda = \{x \in \mathbb{R}^n; g(x) > \lambda\}$. Show that $\int_{\mathbb{R}^n} |f - g| dx = \int_{\mathbb{R}} |(F_\lambda \setminus G_\lambda) \cup (G_\lambda \setminus F_\lambda)|$.

5.(15%) Let μ be a finite measure on X, and for measurable functions f, g on X, let $d(f,g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu$. Show that $\{f_n\}$ converges in measure to f if and only if $\lim_{n\to\infty} d(f_n, f) = 0$.

6.(15%) Assume $\{\phi_n\}$ is an orthonormal system in $L^2([0,1])$. Define

$$E = \{x \in [0,1]; \lim \phi_n(x) \text{exists}\}.$$

Let $\phi(x) = \lim_{n\to\infty} \chi_E \phi_n(x)$. Show that $\phi(x) \equiv 0$.

7.(15%) Assume $f(x)$ is continuous on $\mathbb{R}$, find the limit

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x + \frac{1}{n}) \cos(nx) dx.$$