臺灣大學數學系
107 學年度下學期博士班資格考試題
科目：實分析

2019.02.21

PH.D QUALIFYING EXAM : REAL ANALYSIS 2019 SPRING

1.(15%) Let (X, M, μ) be a measure space. For $1 \leq p < \infty$ show that $f \in L^p(X)$ if and only if $\sum_{n=1}^{\infty} n^{-(p+1)} \mu\{|f| > \frac{1}{n}\} + \sum_{n=1}^{\infty} n^{p-1} \mu\{|f| > n\} < \infty$.

2.(10%) Construct a sequence of sets $E_1, E_2, \cdots, E_k, \cdots$ such that E_k is decreasing to a set E and $|E_k|_e < \infty$ and $\lim_{k\to\infty} |E_k|_e > |E|_e$.

3.(15%) Let $\phi(x)$ be a bounded measurable function in $\mathbb{R}^n$ such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi(x) = 1$. For $\epsilon > 0$ denote $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$. Show that if $f \in L^1$ then for x in the Lebesgue set of f, we have

$$\lim_{\epsilon \to 0} f * \phi_\epsilon(x) = f(x).$$

4.(15%) Let $f(x) \in L^1(\mathbb{R}^n)$, and $Mf(x)$ be its Hardy-Littlewood maximal function. Show that $Mf(x)$ is lower semicontinuous.

5.(15%) Suppose B is a Lebesgue nonmeasurable set in $\mathbb{R}^n$. Show that we can find a subset $B_0 \subset B$ such that B_0 is also nonmeasurable and if $A \subset B_0$ is Lebesgue measurable, then $|A| = 0$.

6.(15%) Let $\{\phi_k\}$ be an orthonormal system and $\{c_k\}$ be a sequence of numbers in ℓ^2. Show that there exists a function $f \in L^2$ such that its Fouerier series with respect to $\{\phi_k\}$ is exactly equal to $\sum c_k \phi_k(x)$.

7.(15%) Let (X, M, μ) be a measure space with $\mu(X) < \infty$. Given a sequence of measurable functions $\{f_n\}$ such that each f_n is finite μ-a.e. Is it possible to find a positive sequence $\{c_n\}$, i.e. $c_n > 0$ for every n, such that $\lim_{n\to\infty} c_n f_n(x) = 0$ μ-a.e. ? Show your result.