臺灣大學數學系
106 學年度下學期博士班資格考試題
科目：實分析 2018.03.01

PH.D QUALIFYING EXAM : REAL ANALYSIS 2018 SPRING

1.(15%) Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f(1) = 2$ and $f'(x) = \frac{1}{x^4+f(x)^2}$. Does the limit $\lim_{x\to\infty} f(x)$ exist ? Prove or disprove your result.

2. Given a continuous function $f(x) : [0,1] \to \mathbb{R}$.

(1).(10%) Show that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$.

(2).(15%) Show that $\lim_{n\to\infty} n\int_0^1 x^n f(x)dx = f(1)$.

3.(15%) Suppose $f \in L^\infty(X)$ on a measure space (X, μ) with $0 < ||f||_\infty$ and $\mu(X) < \infty$. For $p > 0$, let $\beta_p = \int_X |f|^p d\mu$. Show that $\frac{\beta_{p+1}}{\beta_p} \to ||f||_\infty$ when $p \to \infty$.

4.(10%) Let μ be the Lebesgue measure on the σ-algebra M of Lebesgue measurable subsets of $[0,1]$. Let N be a sub-σ-algebra of M. Prove that given a function $f \in L^1([0,1], M, \mu)$, there exists a unique function $g \in L^1([0,1], N, \mu)$ such that $\int_A g d\mu = \int_A f d\mu$ for every $A \in N$.

5. (10%) Let

$$f(x) = \begin{cases} \cos x & \text{if } x \in Q^c \\ \sin x & \text{if } x \in Q \end{cases}$$

Does the Lebesgue integral $\int_0^1 f$ exist ? Prove or disprove it.

6.(15%) Suppose $\{A_n\}$ is a sequence of disjoint measurable sets of $[0,1]$ with $\bigcup A_n = [0,1]$. If $\{B_n\}$ is a sequence of measurable subsets of $[0,1]$ such that $\lim_{n\to\infty} |B_n \cap A_k| = 0$ for all k. Show that $\lim_{n\to\infty} |B_n| = 0$.

7. (10%) Let A, B be two measurable subsets of $\mathbb{R}^1$. Define $f(x) = |(A - x) \cap B|$. Evaluate $\int_{\mathbb{R}^1} f(x)$.