臺灣大學數學系
106 學年度上學期博士班資格考試題
科目：實分析 2017.09.14

PH.D QUALIFYING EXAM : REAL ANALYSIS 2017 FALL

1.(10%) Let Z be a measure zero set in $\mathbb{R}^1$. Show that the set $\{x^2 : x \in Z\}$ has measure zero.

2.

(1).(10%) Suppose $f \in L^s(\mathbb{R}^n) \cap L^t(\mathbb{R}^n)$ for some $1 < s < t < \infty$. Will f belong to $L^p(\mathbb{R}^n)$ for $s < p < t$? Prove or disprove your result.

(2).(10%) For $0 < p < 1$. Show that $||f+g+h||_p \le 3^{\frac{1-p}{p}}(||f||_p + ||g||_p + ||h||_p)$ and $3^{\frac{1-p}{p}}$ is best possible.

3.(15%) Let E be the interval $(0,2)$. Does the Lebesgue integral $\int_E \sin(\frac{1}{x-1})$ exist ? Prove or disprove your result.

4.(15%) Let m be the usual Lebesgue measure on $\mathbb{R}^1$, and define

$$\mu(E) = \int_E \frac{1}{1+x^2} dm(x),$$

for Lebesgue measurable set E. Show that m is absolutely continuous with respect to μ, i.e. show that $m(E) = 0$ whenever $\mu(E) = 0$.

5.

(1).(10%) Let

$$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Find the value of its maximal function at $x = -2$, i.e. what is the value of $f^*(-2)$.

(2).(10%) Denote g^* the Hardy-Littlewood maximal function of g. Given $g \in L^2(\mathbb{R}^n)$, show that $||g^*||_2 \le c||g||_2$ for some constant c only depending on the dimension n.

6.(20%) Given a measurable set $E \subset [0,1]$ with $|E| = \frac{1}{2017}$. Show that E contains an arithmetic progression of length 3, i.e. there are three numbers $a, b, c \in E$ such that $b = a + r$ and $c = a + 2r$ for some $r \neq 0$. (Hint: use Lebesgue Differentiation Theorem)