臺灣大學數學系
105 學年度上學期博士班資格考試題
科目：實分析　　　　2016.09.22

2016 Fall, Real Analysis Qualifying Exam

Each problem counts 20 points.

1. The following two problems are regarding Riemann-Stieltjes integrals.
   (i) Show that if $\int_a^b f dg$ exists, then so does $\int_a^b g df$, and
$$\int_a^b f dg = [f(b)g(b) - f(a)g(a)] - \int_a^b g df.$$
   (ii) For functions f and g defined on the interval $[a, b]$, prove that if f and g have a common discontinuous point, then $\int_a^b f dg$ does not exist.

2. Suppose that f is a continuous function defined on the finite closed interval $[a, b]$. Prove that f is uniformly continuous on $[a, b]$ and the Riemann integral of f over $[a, b]$ exists.

3. Which of the following spaces is (are) separable ?
$$\text{(i)}l^2,\ \text{(ii)}l^\infty,\ \text{(iii)}L^p(\mathbb{R}^2), 1 < p < \infty,\ (iv)L^\infty[0,1].$$
Justify your answer.

4. State and prove the Lebesgue's point theorem for $f(x) \in L^1_{loc}(\mathbb{R}^n)$.

5. Show rigorously the integral
$$\lim_{t\to\infty} \int_0^t \frac{\sin x}{x} = \frac{\pi}{2}.$$