臺灣大學數學系
104 學年度下學期博士班資格考試題
科目：實分析

2016.02.25

1. (20%) Which of the following spaces is (are) separable?

   (i) ℓ^2, (ii) ℓ^∞, (iii) $L^p(\mathbb{R}^2)$, $1<p<\infty$, (iv) $L^\infty[0,1]$.

   Justify your answer.

2. (20%) Construct an absolutely continuous strictly monotone function g on $[0,1]$ such that $g'=0$ on a set of positive measure.

3. (20%) Let $g\in L^\infty[0,1]$ with norm $\|g\|_\infty$. Prove that $\forall\varepsilon>0$, $\exists f\in L^1[0,1]$ such that $\int_0^1 fg\ge\left(\|g\|_\infty-\varepsilon\right)\int_0^1|f|$.

4. (total 20%, 10% each) Suppose $f\in L^1(\mathbb{R}^n)$, $n\ge 2$ and $\int_{\mathbb{R}^n}|f|\,dx\le 1$.

   (i) Prove that $\forall k\in\mathbb{N}$, $\exists r_k\ge 10^k$ such that $\int_{\partial B_{r_k}}|f|\,dS<\frac{1}{r_k}$, where

   $$\partial B_r=\left\{x=(x_1,\cdots,x_n)\in\mathbb{R}^n:\sum_{j=1}^n x_j^2=r^2\right\}\ \text{ for } r>0.$$

   (ii) Suppose that f is continuous on $\mathbb{R}^n$. Prove that $\forall k\in\mathbb{N}$, $\exists r_k\ge 10^k$ and $x_k\in\mathbb{R}^n$ such that $|f(x_k)|\le C\,r_k^{-n}$, where C is a positive constant independent to k

5. (total 20%, 10% each)

   i. Calculate the value $P=\inf\left\{\int_{\mathbb{R}^n}u^2-\ln\left(1+u^2\right)dx:\int_{\mathbb{R}^n}u^{2m}dx=1\right\}$ for $m,n\in\mathbb{N}$. Justify your answer. Discuss all cases of $m,n\in\mathbb{N}$.

   ii. Can the value P be achieved by a function in $L^{2m}(\mathbb{R}^n)$? Justify your answer. Discuss all cases of $m,n\in\mathbb{N}$.