臺灣大學數學系
104 學年度上學期博士班資格考試題
科目：實分析

2015.10.01

Qualified Examination for Real Analysis

10/1/2015

1. (20%) Find a measure zero set C in $\mathbb{R}$ such that C is uncountably infinite. Justify your answer.

2. (20%) Construct a $n-$dimensional $(n \geq 2)$ Cantor set in $\{(x_1, \cdots, x_n): 0 \leq x_i \leq 1, i = 1, \cdots, n\}$ and calculate its measure.

3. (20%) What is the Cantor-Lebesgue function on $[0,1]$? Is the Cantor-Lebesgue function absolutely continuous? Justify your answer.

4. (20%) Suppose that f is nonnegative and measurable on $\mathbb{R}$ and that ω is finite on $(0,\infty)$, where $\omega(\alpha) = |\{x \in \mathbb{R} : f(x) > \alpha\}|$ for $\alpha \geq 0$, and $|\cdot|$ is the Lebesgue measure of $\mathbb{R}$. Assume $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ is finite for some $p > 0$. Calculate $\lim_{a \to 0+} a^p \omega(a)$ and $\lim_{b \to +\infty} b^p \omega(b)$. Justify your answers.

5. (total 20%, 10% each)
   i. Find $f \in L^2(\mathbb{R})$ and $\{f_k\}_{k=1}^\infty$ a sequence of functions in $L^2(\mathbb{R})$ such that $\{f_k\}_{k=1}^\infty$ weakly converges to f in $L^2(\mathbb{R})$ but NOT strongly converges to f in $L^2(\mathbb{R})$. Justify your answer.
   ii. Suppose $\|g_n\|_{L^2(\mathbb{R})} \leq 1$ for $n \in \mathbb{N}$. Prove that there exists a weakly convergent subsequence of $\{g_n\}_{n=1}^\infty$ in $L^2(\mathbb{R})$.