臺灣大學數學系

103 學年度上學期博士班資格考試題

科目：實分析　　2014.0918

Real Analysis Qualifying Exam

Each problem set counts 25 points.

1. Answer the following questions regarding the Riemann-Stieltjes integral.
   (i) Show that if $\int_a^b f dg$ exists, then so does $\int_a^b g df$, and
   $$\int_a^b f dg = [f(b)g(b) - f(a)g(a)] - \int_a^b g df.$$
   (ii) For functions f and g defined on the interval $[a, b]$, prove that if f and g have a common discontinuous point, then $\int_a^b f dg$ does not exist.
2. Show that there exist sets $E_1, E_2, \cdots, E_k, \cdots$, such that
   $$E_k \searrow E, \ |E_k|_l < +\infty,$$
   $$\lim_{k\to\infty} |E_k|_l > |E|_l$$
   with strict inequality, where $|\cdot|_l$ denotes the outer measure. You have to make your argument in details.
3. True or False. For each of the following statements, prove or disprove it.
   (i) If the real function $f(x)$ is differentiable at $x = 0$, f is continuous in certain neighborhood of $x = 0$.
   (ii) Supposed $f : \mathbb{R} \to \mathbb{R}$. If $f'(x)$ is bounded for all $x \in \mathbb{R}$, then $f'(x)$ does not have any jump discontinuity.
   (iii) Let $f(x) \in L^p(E)$ for all $p \geq 1$. Then $f(x) \in L^\infty(E)$.
4.
   (1) State (i) Hardy-Littlewood Lemma for maximal functions and (ii) Simple Vitali (covering) Lemma.
   (2) Prove the Lebesgue differentiation theorem: If $f \in L^1_{loc}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$
   $$\lim_{r\to 0} \frac{1}{B_r(x)} \int_{B_r(x)} f(y)dy = f(x).$$