臺灣大學數學系

102 學年度下學期博士班資格考試題

科目：實分析 2014.02.21

1. (total 30%, 10% each) Let $L=\{f:[0,1]\to\mathbb{R} \text{ is Lebesgue integrable}\}$,

$M=\{f:[0,1]\to\mathbb{R} \text{ is Riemann integrable}\}\subset L$ and

$N=\{f:[0,1]\to\mathbb{R} \text{ is continuous}\}\subset L$ with norm $\|f\|=\int_0^1|f(x)|dx$ for $f\in L$.:

i. Is M a Banach space (complete norm linear space)?

ii. Is the closure of N equal to L?

iii. Does there exist a linear map $T:M\to L$ such that T is onto and bounded (i.e. $\sup\limits_{\substack{f\in M\\ \|f\|=1}}\|T(f)\|<\infty$)?

Justify your answers.

2. (20%) Let $\{f_k\}_{k=1}^{\infty}\subset L[0,1]$ and $f\in L[0,1]$ satisfy

$\lim\limits_{k\to\infty}f_k(x)=f(x),\forall x\in[0,1]$, $\int_0^1|f_k(x)|^2dx<1$ and $\int_0^1|f(x)|^2<\infty$. Prove that

$\int_0^1|f_k-f|^p dx\to 0$ as $k\to\infty$ for any $0<p<2$. What about $p=2$?

3. (20%) Find $\{f_k\}_{k=1}^{\infty}$ a sequence of functions such that f_k converges to f weakly in $L^2[0,1]$, strongly in $L^{3/2}[0,1]$ but not strongly in $L^2[0,1]$.

4. (total 30%, 10% each) Let $\phi_\varepsilon\in C[-1,1]$ satisfy $\forall\varepsilon>0,|\phi_\varepsilon(x)|\le e^{-\frac{1+x}{\varepsilon}}+e^{-\frac{1-x}{\varepsilon}}$ for $x\in(-1,1)$, and $\phi_\varepsilon(x)=x$ at $x=\pm1$. Let $f\in L^1[-1,1]$, $F(x)=\int_{-1}^{x}f(s)ds$ and $n_\varepsilon(x)=\dfrac{e^{\phi_\varepsilon(x)}}{\int_{-1}^{1}e^{\phi_\varepsilon(x)}dx}$ for $x\in[-1,1]$.

i. Find the limit $\lim\limits_{\varepsilon\to0+}\int_{-1}^{1}n_\varepsilon(x)f(x)dx=?$

ii. Find the limit $\lim\limits_{\varepsilon\to0+}\dfrac{1}{\varepsilon}\int_{-1}^{1}\phi_\varepsilon(x)F(x)dx=?$

iii. Prove that $\forall\varepsilon>0,\left|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{1}{\varepsilon}\phi_\varepsilon(x-y)f(y)dydx\right|\le e^{-c_0/\varepsilon}$ for some constant $c_0>0$ independent of ε.