臺灣大學數學系

102 學年度上學期博士班資格考試題

科目：實分析 2013.09.26

Choose 5 from the following 6 problems

1. Let $\{f_k\}$ be a sequence of functions in $L^p, 1 \le p < \infty$, which converges almost everywhere to a function f in L^p. Show that $\{f_k\}$ converges to f in L^p if $||f_k||_{L^p} \to ||f||_{L^p}$.

2. Let S be the set of functions which are positive a.e. on $[0,1]$. Define

$$M_f = \Big(\int_0^1 x^2 f\,dx\Big)\Big(\int_0^1 f^{-1}\,dx\Big), \qquad N_f = \Big(\int_0^1 f\,dx\Big)\Big(\int_0^1 f^{-2}\,dx\Big)^{\frac{1}{2}}.$$

(a) Determine $\inf_{f\in S} M_f$ and find all minimizers of M_f in S.
(b) Determine $\inf_{f\in S} N_f$ and find all minimizers of N_f in S.

3. Let E be a subset of $\mathbb{R}^n$.
(a) Prove that E is Lebesgue measurable if and only if $E = H \setminus Z$, where H is a G_δ set and Z has measure zero. (A set is called G_δ if it is the intersection of a countable collection of open sets.)
(b) Prove that E is Lebesgue measurable if and only if

$$m_*(A) = m_*(E \cap A) + m_*(E^c \cap A) \text{ for every } A \subset \mathbb{R}^n,$$

where m_* dentotes the Lebesgue outer measure.

4. Let $1 \le p \le \infty$, $1 \le q \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$.
(a) Prove that $||f * g||_{L^p} \le ||f||_{L^p}||g||_{L^q}$ if $q = 1$.
(b) Prove that $||f * g||_{L^r} \le ||f||_{L^p}||g||_{L^q}$ if $\dfrac{1}{p} + \dfrac{1}{q} \ge 1$ and $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q} - 1$.

5. Assume $\{x_k\} \subset \mathbb{R}^n$, $c_k \ge 0$, and $\sum_k c_k < \infty$. Let $g(x) = \sum_k c_k f(x + x_k)$.
(a) Show that $g(x)$ is finite almost everywhere if $f(x)$ is Lebesgue integrable on $\mathbb{R}^n$.
(b) Does the same conclusion hold if $f(x) \in L^p(\mathbb{R}^n)$ and $1 < p \le \infty$?

6.
(a) Prove that $(1 - x^2)^b$ is absolutely continuous on $[0,1]$ if $b > 0$.
(b) Let f be a nonnegative absolutely continuous function on $[0,1]$. Is it true that $\sqrt{f}$ is also absolutely continuous on $[0,1]$?