臺灣大學數學系
100 學年度下學期博士班資格考試題
科目：實分析

2012.02.23

Please answer all questions as complete as possible. You will NOT get partial credits for incomplete solutions. Passing grade: 70.

1. True or false:
(a)(5%) Let f be a function on $\mathbb{R}^n$. The set of discontinuities of f is Borel set.
(b)(5%) $|f|$ is measurable $\Rightarrow$ f is measurable.
(c)(10%) Let f be a nonnegative absolutely continuous function on $[a,b]$, then f^s is absolutely continuous for any $0 < s < 1$.

2. A monotone function f on $[a,b]$ is called *singular* if $f' = 0$ a.e.
(a)(5%) Show that any monotone increasing function is the sum of an absolutely continuous function and a singular function.
(b)(5%) Let f be a nondecreasing singular function on $[a,b]$. Then f has the following property: (S) Given $\epsilon > 0, \delta > 0$, there is a finite collection $\{[y_k, x_k]\}$ of nonoverlapping intervals such that

$$\sum |x_k - y_k| < \delta$$

and

$$\sum (f(x_k) - f(y_k)) > f(b) - f(a) - \epsilon.$$

(c)(10%) Let f be a nondecreasing function on $[a,b]$ with property (S) of part (b). Then f is singular.
(d)(10%) Let $\langle f_n \rangle$ be a sequence of nondecreasing singular functions on $[a,b]$ such that the function

$$f(x) = \sum f_n(x)$$

is everywhere finite. Then f is also singular.
(e)(10%) Show that there is a strictly increasing singular function on $[0,1]$.

3. Given any locally integrable function f on $\mathbb{R}^n$. Define the Hardy-Littlewood maximal function

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(r,x)|} \int_{B(r,x)} |f(y)| dy,$$

where $B(r,x)$ is the open ball of radius r centered at x and $|B(r,x)|$ is the measure of $B(r,x)$.

(a)(5%) Show that Mf is measurable.
(b)(5%) For any nonzero integrable f, is Mf integrable?
(c)(10%) Find an example of integrable f, but Mf is not even locally integrable.
(d)(20%) Use any covering lemma you know to prove the Hardy-Littlewood maximal theorem: there exists a constant $C > 0$ such that for any integrable function f we have

$$|\{x : Mf(x) > \lambda\}| \leq \frac{C}{\lambda} \int |f| dx$$

for all $\lambda > 0$.