# 臺灣大學數學系
# 100 學年度上學期博士班資格考試題
# 科目：實分析　　　　2011.09.15

## Real Analysis

Each problem counts 20 points. The full score is 100 points.

1. Prove the following statements.

   (1) Every open set in $\mathbb{R}^1$ can be written as a countable union of disjoint open intervals.

   (2) If $f$ is a bounded increasing function on $[0,1]$, then $f$ has at most a countable number of discontinuities.

2. Suppose that $f$ is a differentiable function with compact support defined on the interval $[0,1]$. Prove that

$$sup_{x\in[0,1]}|f(x)| \leq \|f'\|_{L^2([0,1])}.$$

3.

   (i) If $f$ and $g$ are measurable in $\mathbb{R}^n$, show that the function $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

   (ii) Prove that if $E$ is a measurable set in $\mathbb{R}^{n+m}$, then the set

$$E_x = \{y|(x,y) \in E\}$$

   is measurable in $\mathbb{R}^m$ for almost every $x \in \mathbb{R}^n$.

4. Suppose $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that there is a subsequence $f_{k_j} \to f$ a.e. in $E$.

5. Prove the following Young's convolution theorem: Let $p$ and $q$ satisfy $1 \leq p, q \leq \infty$ and $1/p + 1/q \geq 1$, and let $r$ be defined by $1/r = 1/p + 1/q - 1$. If $f \in L^p(R^n)$ and $g \in L^q(R^n)$, then $f * g \in L^r(R^n)$ and

$$\|f * g\|_{L^r(R^n)} \leq \|f\|_{L^p(R^n)}\|g\|_{L^q(R^n)}.$$