臺灣大學數學系

99 學年度下學期博士班資格考試題

科目：機率論 2011.02.24

請依題號次序作答

Answer questions in order.

1. (10%) State and prove the first and the second Borel-Cantelli lemmas.

2. (10%) State and prove Kolmogorov's inequality.

3. (20%=4+8+8) Let $B_t, t \geq 0$, be the standard one dimensional Brownian motion with $B_0 = 0$.

   (a) Apply the strong law of large numbers to study the almost sure limit of B_n/n as integer $n \to \infty$.

   (b) Apply Kolmogorov's inequality to suitable discrete points in $[n, n+1]$, and then pass to limit to show that

   $$\sum_n P\left(\sup_{t\in[n,n+1]} |B_t - B_n| > n^{0.6}\right) < \infty.$$

   (c) Define a new process $W_t, t \geq 0$, by $W_0 = 0$ and $W_t = tB(1/t)$ for $t > 0$. Show that W_t is also a Brownian motion. (Hint. Use the result in (b) to study the continuity of W_t at $t = 0$.)

4. (10%) Let $U_1, U_2, \ldots$ be a sequence of independent random variables that are uniformly distributed over $[0, 1]$. For each n, let Z_n be the median of the values of $U_1, U_2, \ldots, U_{2n+1}$. That is, if we order $U_1, \ldots, U_{2n+1}$ in increasing order, then Z_n is the $(n+1)$st element in this ordered sequence. Show that the sequence $Z_n \to c$ in probability for some constant c. What is this constant c?

5. (16%=8+8)

   (a) Show that if random variables $Y_n \Rightarrow c$, where c is a constant, then $Y_n \to c$ in probability.

   (b) Suppose that $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant, then $X_n + Y_n \Rightarrow X + c$.

6. (16%=8+8) Let M_n be a supermartingale with respect to the filtration $\mathcal{F}_n$.

   (a) Let ψ be an increasing concave function with $E[|\psi(M_n)|] < \infty$ for all n. Show that $\psi(M_n)$ is a supermartingale with respect to $\mathcal{F}_n$.

   (b) Suppose that $M_n \geq V$ for all n for some random variable V with $E[|V|] < \infty$. Show that M_n converges a.s.

7. (18%=12+6) Let X_n be an irreducible Markov chain. Answer the following questions. No proof is needed.

   (a) Write down the definitions that a state is **transient, null recurrent** and **positive recurrent**, respectively. In which case that the chain has a stationary distribution? Is the stationary distribution unique when exists?

   (b) Give three examples of irreducible Markov chain which is transient, null recurrent and positive recurrent, respectively.