臺灣大學數學系

九十七學年度上學期博士班資格考試題

科目：機率論 2008.09.19

NTU Probability Theory, 2008 PhD Qualify, 2008.9

Problems 1–4, 15 points each; Problems 5,6, 20 points each

1. Let X be a symmetric r.v. such that, for $x > 1$, the tail distribution has the decay $P\{|X| > x\} \approx x^{-\alpha}$. Discuss and prove the integrability of $|X|$, in cases $0 < \alpha \leq 1$ and $1 < \alpha < 2$. Discuss the possibility of $\alpha = 2$.

2. Let Z_λ be a Poisson distribution with mean λ. Prove that $(Z_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to $N(0,1)$, as $\lambda \to \infty$. May consider the case λ be positive integer firstly and then go to the general case.

3. Let $(X_n, \mathcal{F}_n)$ be a martingale, and each X_n is in $L^2(dP)$. The difference is $\xi_{m,n} := X_n - X_m, m < n$. Simplify $E[\xi_{m,n}^2|\mathcal{F}_m]$ to a manageable form, and prove that, if $\sum_n E\xi_{n-1,n}^2 < \infty$, then the martingale convergence holds both a.s. and in mean square.

4. Consider a simple branching model in which we start with one single particle, at time 0. After one unit time, it branches out a random number of sub-particles; each sub-particle then branches independently and with the same offspring distribution(branching mechanism) as the original one, $\{p_k, k \geq 0\}$. Let Z_n denote the population number of particles at time n. Write Z_n in its recursive form and prove that it is a Markov chain.

5. Let X_n (discrete time) be a finite-states irreducible aperiodic MC. 1. what means "irreducible aperiodic" ? 2. assume that X_n is also doubly stochastic, write and prove its unique stationary distribution. 3. how about the 2. if it is the infinite-states.

6. Let $B_t, t \geq 0$, denote the standard Brownian Motion on the line. For any $\theta > 0$, find and prove suitable non-random $\phi_1(t)$ and $\phi_2(t)$ so that both processes $\theta B^2(t) - \phi_1(t)$ and $e^{\theta B_t} - \phi_2(t)$, $t \geq 0$, form martingales w.r.t. the natural filtration.