國立臺灣大學數學系
九十五學年度博士班資格考試試題
科目：機率論

All random variables are given on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

1. (30 %) Let $X, X_n, n \geq 1$, be random variables such that $X_n \to X$ *in probability* as $n \to \infty$.

   (a) Suppose that $|X_n| \leq U$ for some U satisfying $\mathbf{E}[U] < \infty$. Show that $\lim_{n\to\infty} \mathbf{E}[|X_n - X|] = 0$.

   (b) State the definition that the family $\{X_n, n \geq 1\}$ is **uniformly integrable**.

   (c) Show that $\lim_{n\to\infty} \mathbf{E}[|X_n - X|] = 0$ if the sequence $\{X_n, n \geq 1\}$ is uniformly integrable.

2. (30 %) (a) For any random variable Z show that

$$\sum_{n=1}^{\infty} \mathbf{P}(|Z| \geq n) \leq \mathbf{E}[|Z|] \leq \sum_{n=0}^{\infty} \mathbf{P}(|Z| \geq n).$$

   Here $\mathbf{E}[|Z|]$ can be infinite.

   (b) State the (first and the second) Borel-Cantelli Lemma concerning the probabilities $\mathbf{P}(A_n \text{ i.o. })$ and $\sum \mathbf{P}(A_n)$, where $A_n \in \mathcal{F}, n \geq 1$, and i.o. means infinitely often. No proof is needed.

   (c) Let $X_n, n \geq 1$, be a sequence of independent, identically distributed (i.i.d.) random variables with $\mathbf{E}[X_1] = 0$. Let $S_n = X_1 + \cdots + X_n$ and $p > 0$. If $S_n/n^{1/p} \to 0$ almost surely (a.s.), prove that $\mathbf{E}[|X_1|^p] < \infty$.
   Hint. Assume $\mathbf{E}[|X_1|^p] = \infty$. Use the results in (a) and (b) to yield a contradiction.

3. (10 %) Let $X_n, n \geq 1$, be a sequence of i.i.d. random variables with $\mathbf{E}[X_1] = \mu$ and $\mathbf{Var}(X_1) = \sigma^2 \in (0, \infty)$. Let $S_n = X_1 + \cdots + X_n$. Determine the constants C, α, β, γ such that $Cn^\alpha \mu^\beta \sigma^\gamma \left(\sqrt[3]{S_n} - \sqrt[3]{n\mu}\right)$ converges weakly to a standard normal distribution as $n \to \infty$.

4. (30 %) Let $\xi_j^n, j, n \geq 1$, be i.i.d. nonnegative integer-valued random variables. Put $X_0 = 1$ and define $X_n, n \geq 1$, as follows: $X_n = \xi_1^n + \cdots + \xi_{X_{n-1}}^n$ if $X_{n-1} > 0$, and $X_n = 0$ if $X_{n-1} = 0$. Denote $\mathcal{F}_n = \sigma(\xi_j^m, j \geq 1, 1 \leq m \leq n)$ and $\mu = \mathbf{E}[\xi_1^1] < \infty$.

   (a) Show that $M_n = X_n/\mu^n$ is a martingale with respect to $\mathcal{F}_n$.

   (b) State a martingale convergence theorem without proof. Apply this convergence theorem to M_n to solve (c).

   (c) Suppose that $\mu < 1$. Find the limit of M_n and X_n. Is $\{M_n, n \geq 1\}$ uniformly integrable? Does M_n converge a.s.? Does M_n converge in L_1? Give detailed reasons to your answers.