臺灣大學數學系111學年度第1學期博士班一般資格考試

科目：機率論

2022.09.12

1. ($15\% = 5+5+5$) Let (X_n) and (Y_n) be two sequences of random variables. Also let X and Y be two random variables. Does the following hold?

   (a) If $X_n \to X$ almost surely and $Y_n \to Y$ almost surely then $X_n + Y_n \to X + Y$ almost surely.

   (b) If $X_n \to X$ in probability and $Y_n \to Y$ in probability then $X_n + Y_n \to X + Y$ in probability.

   (c) If $X_n \to X$ in distribution and $Y_n \to Y$ in distribution then $X_n + Y_n \to X + Y$ in distribution.

   Prove your assertions.

2. ($18\% = 9+9$) Let (X_n) be a sequence of independent random variables with $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 < \infty$ for all n. For $n \geq 1$, set $S_n = X_1 + \cdots + X_n$ and define $M_n = \max\{|S_1|, \ldots, |S_n|\}$.

   (a) Prove the Kolmogorov maximal inequality: for any $a > 0$,

   $$\mathbb{P}(M_n \geq a) \leq \frac{1}{a^2}\operatorname{Var}(S_n).$$

   (b) Prove that if $\sum_{n=1}^{\infty} \operatorname{Var}(X_n) < \infty$ then $\sum_{n=1}^{\infty} X_n$ converges almost surely.

3. (15%) For each $n \in \mathbb{N}$, let $B_{1,n}, B_{2,n}, \ldots, B_{n,n}$ be i.i.d. Bernoulli($1/n$) random variables, and let $(G_{i,j})_{i\leq j}$ be i.i.d. standard normal random variables. Further assume that $(B_{i,j})_{i\leq j}$ and $(G_{i,j})_{i\leq j}$ are independent. Define, for $n \in \mathbb{N}$,

   $$X_n = \sum_{i=1}^{n} B_{i,n} G_{i,n}.$$

   Prove that (X_n) converges in distribution to a product $\sqrt{N}G$, where N is a Poisson random variable with parameter 1, G is standard Gaussian, and N and G are independent.

4. ($15\% = 5+5+5$) Let (X_n) be a sequence of random variables in L^2, and let (δ_n) be a sequence of positive real numbers. We say that (X_n) has fluctuations of order at least δ_n if there exist positive constants $c_1, c_2 > 0$ such that for all large n, and for any real numbers $a \leq b$ with $b - a \leq c_1\delta_n$, one has

   $$\mathbb{P}(a \leq X_n \leq b) \leq 1 - c_2.$$

   (a) Suppose that (Y_n) is a sequence of i.i.d. random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$. Set $S_n = Y_1 + \cdots + Y_n$. Show that (S_n) has fluctuations of order at least $\sqrt{n}$.

(b) i. Show that if (X_n) has fluctuations of order at least δ_n, then

$$\liminf_{n\to\infty} \frac{\operatorname{Var}(X_n)}{\delta_n^2} > 0.$$

ii. Does the converse of i. hold in general? Give a proof if the converse is true; otherwise provide a counterexample.

5. (15% = 10 + 5) Let $X_n = \xi_1 + \cdots + \xi_n$, where (ξ_i) is a sequence of i.i.d. Rademacher random variables (that is, $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$ for all $i \geq 1$). Let $\tau = \min\{n \geq 1 : X_n = 1\}$ and define $Y_n = X_{\min\{\tau,n\}}$.

   (a) Show that (Y_n) is a martingale with $\sup_n \mathbb{E}|Y_n| < \infty$, and that (Y_n) converges almost surely.

   (b) Is it true that

$$\lim_{n\to\infty} \mathbb{E}Y_n = \mathbb{E} \lim_{n\to\infty} Y_n?$$

   Prove your assertion.

6. (22% = 10 + 4 + 4 + 4)

   (a) Let (B_t) be a standard Brownian motion. Show that (B_t) is jointly measurable. That is, the map $(t, \omega) \mapsto B_t(\omega)$ from $[0, \infty) \times \Omega \to \mathbb{R}$ is measurable relative to the product σ-algebra $\mathcal{B} \times \mathcal{F}$, where Ω is the sample space on which we define the Brownian motion, $\mathcal{F}$ is the corresponding σ-algebra, and $\mathcal{B}$ is the Borel σ-algebra on $[0, \infty)$.

   (b) Define the zero set of Brownian motion

$$Z = \{t \geq 0 : B_t = 0\}.$$

   Show that Z is (i) unbounded, (ii) nowhere dense, and (iii) has Lebesgue measure zero.