# 臺灣大學數學系
# 110 學年度上學期博士班資格考試題
## 科目：機率論

2021.09.24

Answer the following questions <u>in order</u>. 請依序作答，預留足夠空間給未完成或跳過的題目。

1. (30%=18+12) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables on probability space $(\Omega, \mathscr{F}, \mathbf{P})$.
   (a) Consider the following property (P) and statement (S) :
   (P) For every subsequence $X_{n(m)}$ there exists a further subsequence $X_{n(m_k)}$ that converges almost surely (a.s.) to $Z$
   (S) The sequence $\{X_n\}_{n=1}^{\infty}$ satisfies property (P) if and only if $X_n \to Z$ *in certain mode of convergence* as $n \to \infty$.

   Specify the *mode of convergence* in (S) and then prove the statement (S). If you want to apply any particular facts/lemmas/theorems, you have to state and prove them first.
   (b) Suppose the random variables $X_n$'s are independent and identically distributed (i.i.d.) with mean zero and $\mathbf{E}[X_1^4] < \infty$. Show that $S_n/n^{4/5} \to 0$ a.s., where $S_n = X_1 + \cdots + X_n$. Based on the approach of your proof, can you furthermore improve the factor $n^{4/5}$?

2. (30%=4+10+4+12) Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of nonnegative random variables. Suppose that $U_n$ converges weakly/in distribution to $V$ (written $U_n \Rightarrow V$).
   (a) Show that $\mathbf{P}(V \geq 0) = 1$.
   (b) Answer the following two questions by giving proofs or counterexamples.
   - Does it hold in general that $\mathbf{E}[V] < \infty$?
   - If $\mathbf{E}[V] < \infty$, does it hold that $\mathbf{E}[U_n] \to \mathbf{E}[V]$ in this case?

   (c) Write down the definition that a collection of random variables $W_\lambda, \lambda \in \Lambda$, is "uniformly integrable".
   (d) Suppose that the sequence $\{U_n\}$ is uniformly integrable. Show that $\mathbf{E}[V] < \infty$ and $\mathbf{E}[U_n] \to \mathbf{E}[V]$ as $n \to \infty$.

3. (24%=12+10+2) Let $M_n, n = 0, 1, 2, \ldots$, be a submartingale with respect to the filtration $\mathscr{F}_n$ and $\tau$ be a stopping time satisfying $\mathbf{P}(\tau \leq k) = 1$ for some positive integer $k$.
   (a) Prove that $\mathbf{E}[M_0] \leq \mathbf{E}[M_\tau] \leq \mathbf{E}[M_k]$. You might need to show that $M_{\tau \wedge n}$ is also a submartingale.
   (b) For $\lambda > 0$, let $A = \left\{ \max_{0 \leq \ell \leq n} M_\ell^+ \geq \lambda \right\}$. Prove Doob's inequality :
   $$\lambda \mathbf{P}(A) \leq \mathbf{E}[M_n 1_A] \leq \mathbf{E}[M_n^+],$$
   where $1_A$ is the indicator function of $A$ and $M_\ell^+$ is the positive part of $M_\ell$.
   (c) Describe a possible application of Doob's inequality. No proof is needed.

4. (16%=8+8) Let $Y_n$ be a discrete time irreducible recurrent Markov chain with period $d$. Denote $P_{ij}^n$ be the $n$-step transition probability of the chain from state $i$ to state $j$.
   (a) Explain the terms "irreducible", "recurrent", "transient", and "period $d$".
   (b) For simplicity, suppose that $d = 1$. It can be shown that $\lim_{n\to\infty} P_{ii}^n = \mu_i$ exists for every $i$. By assume this fact derive all the subsequent results. In particular, show that if the chain has only finitely many states then $Y_n$ has a unique stationary distribution and thus is positive recurrent.