臺灣大學數學系

107 學年度下學期博士班資格考試題

科目：機率論 2019.02.21

1. (24%) (a) (8%) Let $\{X_n\}_{n=1}^{\infty}$, X, Y be random variables. Suppose X_n converges to Y in distribution and X_n converges to X almost surely. Prove that X has the same distribution as Y.

   (b) (8%) Suppose that X_n has a gamma distribution with parameters a_n and 1. Here $\lim_{n\to\infty} a_n = a > 0$. (Note that the density function of X_n is $x^{a_n-1}\exp(-x)/\Gamma(a_n)$ for $x > 0$.) Prove that X_n converges in distribution to the gamma distribution with parameters a and 1.

   (c) (8%) Suppose that $\sum_{k=1}^{\infty} c_k < \infty$ with $c_k > 0$ for all k. Suppose that Z_k has a gamma distribution with parameters c_k and 1 for all k and that the Z_k's are independent. Prove that $S_n = \sum_{k=1}^{n} Z_k$ converges almost surely and find the distribution of the limit.

2. (16%) Assume that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed exponential random variables with density function $\exp(-x)$ for $x > 0$. Denote the maximum of $X_1, \cdots, X_n$ by $X_{(n)}$

   (a) (8%) Prove that $X_{(n)}/\ln(n)$ converges to 1 in probability.

   (b) (8%) Prove that

$$\liminf_{n\to\infty} \frac{X_{(n)}}{\ln(n)} \geq 1, \quad \text{almost surely.}$$

3. (16%) Let $Y_1, Y_2, \cdots$ be independent and identically distributed with the density $f(y)$. When $|y| > 1$, $[2\alpha/(\alpha-1)]f(y) = |y|^{-\alpha}$ for some $\alpha > 1$. Otherwise, $[2\alpha/(\alpha-1)]f(y) = 1$. Let $X_k = Y_k/k$ for all k, and let $S_n = \sum_{k=1}^{n} X_k$. Show that S_n converges almost surely (to a finite, possibly random limit) if and only if $\alpha > 2$.

4. (16%) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider the following definition.

   Definition: A set $A \subseteq \Omega$ is called negligible with respect to $(\Omega, \mathcal{F}, P)$ if there exists $B \in \mathcal{F}$ such that $A \subseteq B$ and $P(B) = 0$.

   Let $\mathcal{N}$ denote the class of all negligible sets with respect to $(\Omega, \mathcal{F}, P)$.

   (a) (5%) Is $\mathcal{N}$ a π-system? Explain.

   (b) (3%) Write a definition of a complete probability space.

   (c) (5%) Show that $\mathcal{F}$ is a σ-field of subsets of Ω

   (d) (3%) Part (c) implies that $(\Omega, \mathcal{F}, P)$ is a complete probability space. In fact, it holds that $\mathcal{F}$ is the smallest σ-field of subsets of Ω such that $\mathcal{F} \subset A$ and $(\Omega, \mathcal{F}, P)$ is complete. Suppose that $\mathcal{G}$ is a field of subsets of Ω, and that $\sigma(\mathcal{G}) = \mathcal{F}$. Let Q be a probability measure on the space $(\Omega, \mathcal{F})$ such that for all $A \in \mathcal{G}$ it holds that $Q(A) = P(A)$. Explain why $Q(A) = P(A)$ for $A \in \mathcal{F}$.

5. (15%) Consider the following statement

$$X_n \to X \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i \to X.$$

Indicate whether this statement is true or false, providing proof or counter example accordingly, for each of the following modes of convergence.

(a) (5%) almost sure,

(b) (5%) L^p, $1 \le p < \infty$.

(c) (5%) in probability.

You may use the following fact: if $\{x_n\}_{n\ge1}$ is a sequence of numbers such that $\lim_n x_n$ exists and is equal to x, then $\lim_n \frac{1}{n}\sum_{i=1}^{n} x_i = x$.

6. (13%) Suppose $X_1, X_2, \ldots$ be independent Poisson random variables with $E(X_i) = \lambda_i$. Namely, $P(X_i = k) = \lambda_i^k e^{-\lambda_i}/k!$ for $k = 0, 1, 2, \ldots$. Let $S_n = \sum_{i=1}^{n} X_i$. Show that $S_n/E(S_n)$ converges to 1 almost surely, if $\sum_n \lambda_n = \infty$.