臺灣大學數學系

102 學年度上學期博士班資格考試題

科目：機率論 2013.09.27

Answer the following questions in order.

1. (25%=10+5+10)

   (a) Give a probabilistic proof, using weak law of large numbers, of Weierstrass polynomial approximation theorem.

   For (b) and (c), let (F_n) be a sequence of distribution functions such that the limit $\lim_{n\to\infty} \int x^k \, dF_n(x) = m_k$ exists for each $k = 0, 1, 2, \ldots$

   (b) Show that (F_n) is tight.

   (c) Suppose that G is a distribution function concentrated on a compact set of $\mathbb{R}$ and $\int x^k \, dG(x) = m_k, k = 0, 1, 2, \ldots$ Show that $F_n \Rightarrow G$. (Hint. Apply approximation theorem.)

2. (25%=5+10+10) Let $B_t = B(t), t \geq 0$, be a one dimensional Brownian motion starting at 0. Define a new process $W_t, t \geq 0$, by $W_0 = 0$ and $W_t = tB(1/t)$ for $t > 0$.

   (a) Study the almost sure limit of B_n/n as the integer $n \to \infty$.

   (b) Derive the estimate

   $$P\left(\sup_{t\in[n,n+1]} |B_t - B_n| > n^{2/3}\right) \leq c\, n^{-4/3}$$

   for some constant $c > 0$. Hint. Consider $P\left(\sup_{0<k\leq 2^m} \left|B\left(n + \frac{k}{2^m}\right) - B(n)\right| > n^{2/3}\right)$ first.

   (c) Use (b) to show that W_t is continuous at $t = 0$ almost surely. Then verify that the process W_t is also a Brownian motion.

3. (25%=10+5+10) Let Z_n be a martingale and τ be a stopping time.

   (a) Suppose that $P(\tau \leq k) = 1$ for some integer $k \in \mathbb{N}$. Show that $E[Z_0] = E[Z_\tau] = E[Z_k]$.

   (b) Give an example of Z_n such that $E[Z_0] > E[Z_\tau]$ for some unbounded τ.

   (c) Show that $E[Z_0] = E[Z_\tau]$ if $P(\tau < \infty) = 1, E[|Z_\tau|] < \infty$, and $E[Z_n 1_{\{\tau > n\}}] \to 0$ as $n \to \infty$.

4. (25%=9+6+10) Let Y_n be an irreducible Markov chain on a countable state space S. No proof is needed in (b).

   (a) Give the definition that a state $x \in S$ is **transient, null recurrent** and **positive recurrent**, respectively.

   (b) When does the chain have a stationary distribution? Describe this distribution when exists.

   (c) Discuss the transience or null/positive recurrence of the symmetric simple random walk on $\mathbb{Z}^d, d = 1, 2, 3, \ldots$ Give an outline of the proof.