臺灣大學數學系
100 學年度下學期博士班資格考試題
科目：機率論 2012.02.23

1. (12%) Suppose that $X_1, X_2, \cdots, X_n$ are independent and identically distributed with uniform distribution on the integers $\{1, 2, \ldots, N\}$ with $n < N$ and let

$$T_N = \min\{k : \text{there exists a } j < k \text{ such that } \{X_j = X_k\}\}.$$

   (a) (8%) Find a suitable constant sequence $\{b_N\}$, and show that, as N goes to infinity, T_N/b_N converges in distribution to a limit Y_∞ with distribution F.

   (b) (4%) Describe $F(\cdot)$.

2. (15%) Consider the following statement

$$X_n \to X \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i \to X.$$

   Indicate whether this statement is true or false, providing proof or counterexample accordingly, for each of the following modes of convergence.

   (a) (5%) almost sure,

   (b) (5%) L^p, $1 \le p < \infty$.

   (c) (5%) in probability.

   You may use the following fact: if $\{x_n\}_{n\ge1}$ is a sequence of numbers such that $\lim_n x_n$ exists and is equal to x, then $\lim_n \frac{1}{n}\sum_{i=1}^{n} x_i = x$.

3. (13%) Suppose $X_1, X_2, \ldots$ be independent Poisson random variables with $E(X_i) = \lambda_i$. Namely, $P(X_i = k) = \lambda_i^k e^{-\lambda_i}/k!$ for $k = 0, 1, 2, \ldots$. Let $S_n = \sum_{i=1}^{n} X_i$. Show that $S_n/E(S_n)$ converges to 1 almost surely, if $\sum_n \lambda_n = \infty$.

4. (20%) Let $\{(X_k, Y_k), k \ge 1\}$ be a sequence of independent identically distributed random vectors (i.e., (X_j, Y_j) is independent of (X_k, Y_k) for $j \ne k$, and the distribution of (X_j, Y_j) does not depend on j). Suppose that each X_k and Y_k takes values in the set $\{\cdots, -1, 0, 1, 2, \cdots\}$. Suppose further that $E(X_1) = E(Y_1) = 0$ and $E(X_1Y_1) = c$, and that X_1 and Y_1 have finite non-zero variances. Let U_0 and V_0 be positive integers, and define $(U_{n+1}, V_{n+1}) = (U_n, V_n) + (X_{n+1}, Y_{n+1})$ for each $n \ge 0$. Let $T = \min\{n : U_nV_n = 0\}$ be the first time that the random walk on the plane, (U_n, V_n), hits one of the axes.

   (a) (6%) Show that $U_nV_n - cn$ is a martingale.

   (b) (6%) State optional stopping theorem. Is T a stopping time? Can the optional stopping theorem be applied to the martingale in (a) to find $E(T)$? Explain.

   (c) (8%) Let $T_m = T \wedge m$. Show that $E(T_m) = c^{-1}[E(U_{T_m}V_{T_m}) - U_0V_0]$. Argue that, subject to an interchange of limits and expectations, $E(T) = -U_0V_0/c$. (If $c < 0$, the interchange can be formally justified by showing that $E(T) < \infty$, though you are not asked to investigate this here. If $c > 0$ then this reasoning leads to an absurdity and we infer that, in this case, $E(T) = \infty$.)

5. (20%) Let B_t, $t \geq 0$, be the standard one-dimensional Brownian motion. Levy's modulus of continuity is defined as

$$osc(\delta) = \sup\{|B_s - B_t| : s, t \in [0,1], |t-s| < \delta\}.$$

(a) (8%) Let $\Delta_{m,n} = \sup\{|B_t - B(m2^{-n})| : t \in [m2^{-n}, (m+1)2^{-n}]\}$ where m and n are natural numbers. Show, for $a \geq 1$,

$$P(\Delta_{m,n} \geq a2^{-n/2}) \leq 4\exp(-a^2/2).$$

(b) (12%) Use (a) to show that with probability 1,

$$\limsup_{\delta \to 0} osc(\delta)/(\delta \log(1/\delta))^{1/2} \leq 6.$$

6. (20%) Consider a mobile radio that is moving on the integer points of the real line according to a random walk. Let $S(n)$ denote the position of the mobile radio at time instant n and define $S(n)$ as follows: $S(0) = 0$ and

$$S(n+1) = \begin{cases} S(n)+1, & \text{with probability } p, \\ S(n)-1, & \text{with probability } 1-p. \end{cases}$$

Let $Y(n) = |S(n)|$.

(a) (8%) Determine

$$P(S(n) = i | Y(n) = i, Y(n-1), Y(n-2), \ldots, Y(1)).$$

(b) (12%) Show that $Y(n)$ is also a Markov chain and determine its probability transition matrix.
Hint: You can use (a) to compute

$$P(Y(n+1) = i+1 | Y(n) = i, Y(n-1), Y(n-2), \ldots, Y(1))$$

and

$$P(Y(n+1) = i-1 | Y(n) = i, Y(n-1), Y(n-2), \ldots, Y(1)).$$