臺灣大學數學系
100 學年度上學期博士班資格考試題
科目：機率論

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請依題號次序作答

Answer questions in order.

1. (10%) Let $Z_1, Z_2, \dots$ be independent and identically distributed (i.i.d.) uniform random variables on $\{1, 2, \dots, n\}$, and $T_n = \inf\{k : |\{Z_1, Z_2, \dots, Z_k\}| = n\}$ be the first time in the sequence $Z_1, Z_2, \dots$ that all numbers $\{1, 2, \dots, n\}$ are present. Find a suitable constant sequence (b_n), and show that T_n/b_n converges in L_2 and thus in probability. Determine the limit.

2. (30%=12+6+6+6) Let $X_1, X_2, \dots$ be independent random variables that each one has mean zero and finite variance, and $S_n = X_1 + \cdots + X_n$.

   (a) For $a > 0$, prove Kolmogorov's maximal inequality:

   $$\mathbf{P}\left(\max_{1\le k\le n} |S_k| \ge a\right) \le \frac{\mathbf{Var}(S_n)}{a^2}.$$

   (b) Let $\rho_k = \sup_{m,n\ge k} |S_m - S_n|$ and $\varepsilon > 0$. Apply the above inequality to show that

   $$\mathbf{P}(\rho_k > \varepsilon) \le \mathbf{P}\left(\sup_{n\ge k} |S_n - S_k| > \varepsilon/2\right).$$

   (c) Conclude from (b) that $X_1 + X_2 + \cdots$ converges almost surely if $\sum_{n=1}^{\infty} \mathbf{Var}(X_n) < \infty$.

   (d) Suppose further that the random variables $X_1, X_2, \dots$ are identically distributed. For any $\delta > 0$ show that $\dfrac{S_n}{n^{1/2}(\ln n)^{\frac{1}{2}+\delta}} \to 0$ almost surely as $n \to \infty$. (Hint. You might need Kronecker's lemma.)

3. (14%=10+4)

   (a) Let Y_λ be a random variable having Poisson distribution with parameter λ. Find constants $a(\lambda)$ and $b(\lambda)$, and show that $(Y_\lambda - a(\lambda))/b(\lambda)$ converges weakly as $\lambda \to \infty$. Determine the limit distribution.

   (b) Let $U_1, U_2, \dots$ be i.i.d. Poisson random variables with a fixed parameter λ and let $a > \lambda$. Evaluate the limit $\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(U_1 + \cdots + U_n \ge na)$ if exists.

4. (14%=10+4) Let $(\mathbf{P}_{xy})$ be the transition probabilities of a Markov chain X_n on state space $\mathbb{S}$, and $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$. Call a function *harmonic* (*super-harmonic, sub-harmonic*) at x if $\sum_{y\in\mathbb{S}} \mathbf{P}_{xy} f(y) = f(x)$ ($\le, \ge f(x)$, respectively.) For $A \subset \mathbb{S}$, define the stopping time $\tau_A = \inf\{n \ge 0 : X_n \in A\}$ that is the first entrance time into A. For any bounded function f on $\mathbb{S}$ denote

   $$\begin{aligned} (Sf)(x) &= \sum_y \mathbf{P}_{xy} f(y) = \mathbf{E}_x[f(X_1)], \qquad (If)(x) = f(x), \\ Z_n &= f(X_n) - f(X_0) - \sum_{k=0}^{n-1} ((S-I)f)(X_k), n \ge 1,\ Z_0 = 0, \\ U_A(x) &= \mathbf{P}_x(\tau_A < \infty). \end{aligned}$$

   (a) Prove that Z_n is a martingale with respect to $\mathcal{F}_n$.

   (b) Show that U_A satisfies $0 \le U_A \le 1$, $U_A = 1$ on A, and $(S - I)U_A = 0$ on A^c.

5. (16%=10+6)

(a) Given a Markov chain, write down the definitions that a state is **transient, null recurrent** and **positive recurrent**, respectively. In which case(s) that the chain has a stationary distribution? Is the stationary distribution unique when exists?

(b) Use the results in Question 4 to show that an irreducible Markov chain is recurrent if and only if every nonnegative super-harmonic function is constant.

6. (16%=12+4) Let $B(t), t \geq 0$, be the standard one dimensional Brownian motion.Fix $t > 0$ and let $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$.

(a) Apply Borel-Cantelli lemma to show that $\sum_{m=1}^{2^n} \Delta_{m,n}^2$ converges almost surely as $n \to \infty$. Determine the limit.

(b) What conclusion can you get from (a)?