臺灣大學數學系

99 學年度上學期博士班資格考試題

科目：偏微分方程　　2010.09.17

1. Solve the equations.

(a) $u_x + yu_y = \frac{(x+1)^2}{x^2+1}u, u(0,y) = y;$

(b) $u_x u_y = 2xu, u(1,y) = 2y.$

2. (Poincare type inequality) Let u be a C^1 function defined on $\mathbb{R}$.

(a) Assume $u(0) = u(1) = 0$. Prove that there exists a constant C which is independent of u such that

$$\int_0^1 |u(x)|^2\,dx \le C\int_0^1 |u_x(x)|^2\,dx.$$

(b) Assume $\int_{-\infty}^{\infty} e^x|u(x)|^2\,dx < \infty$. Prove that there exists a constant C which is independent of u such that

$$\int_{-\infty}^{\infty} e^x|u(x)|^2\,dx \le C\int_{-\infty}^{\infty} e^x|u_x(x)|^2\,dx.$$

What is the best constant C?

3. Let u be a smooth function in $\mathbb{R}^3$.

(a) Solve the problem

$$u_{xx} = u_{yy} = u_{zz},$$
$$u(x,0,0) = x^2 + x, u_y(x,0,0) = 0, u_z(x,0,0) = 0, u_{yz}(x,0,0) = 1.$$

Is the solution unique?

(b) Is it true that if $u(x,y,z)$ satisfies $u_{xx} = u_{yy} = u_{zz}$, then it has the form

$$u = F(x+y+z) + G(x+y-z) + P(x-y+z) + Q(x-y-z).$$

for some functions $F(t), G(t), P(t)$ and $Q(t)$?

4. Suppose $u(x,t)$ is a smooth solution of

$$\begin{aligned} u_t - \Delta u + c(x)u &= 0 \text{ in } U\times(0,\infty) \\ u &= 0 \text{ on } \partial U\times[0,\infty), \\ u(x,0) &= g(x), \end{aligned}$$

where U is a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂U.

(a) Show that $\int_U(|\nabla u|^2 - c(x)u^2)\,dx$ is nonincresing in t.

(b) Show that $u \ge 0$ if $g \ge 0$.

(c) Show that $u(x,t)$ is nondecresing in t if $\Delta g - c(x)g \ge 0$. (Hint: Consider $v = u(x,t+a) - u(x,t)$. Prove that $v \ge 0$ for each $a > 0$.)

5. Let u be a harmonic function in $\mathbb{R}^2$. Prove that

$$u(0) = \frac{1}{2\pi R}\int_{|x|=R} u(x)\,dS \quad\text{and}\quad |\nabla u(0)| \le \frac{C}{R}\max_{|x|=R}|u(x)|$$

for some constant C independent of u.