臺灣大學數學系
98 學年度下學期博士班資格考試題
科目：偏微分方程

2010.02.25

1. (20 points) Suppose $u(t) \in C^2(\mathbb{R}^+)$ satisfies

$$\begin{cases} \dfrac{d^2}{dt^2}u + \dfrac{N}{t}\dfrac{du}{dt} - \dfrac{N}{t^2}u \geq 0, \text{ in } \mathbb{R}^+, \\ \lim\limits_{t\to\infty} \dfrac{u(t)}{t} = 0, \\ u(t) > 0, \quad \forall t > 0, \end{cases}$$

where N is a positive integer. Show that, for $0 < T \leq t < \infty$, u satisfies

$$u(t) \leq (\frac{t}{T})^{-N} u(T).$$

Hint: Find the fundamental solutions for the equation

$$\frac{d^2}{dt^2}u + \frac{N}{t}\frac{du}{dt} - \frac{N}{t^2}u = 0, \text{ in } \mathbb{R}^+,$$

and apply the maximum principle. Make sure you use the decay condition of $u(x)$ when you make the argument.

2. (20 points) Integrate by parts to prove:

$$\int_U |Du|^p dx \leq C(\int_U |u|^p dx)^{\frac{1}{2}} (\int_U |D^2u|^p)^{\frac{1}{2}}$$

for $2 \leq p < \infty$ and all $u \in W^{2,p}(U) \cap W_0^{1,p}(U)$.

3. (20 points) Solve the equation

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 \text{ for } x_2 > 0, \\ u(x_1, 0) = -e^{x_1^2}. \end{cases}$$

4. (20 points) Suppose u is a smooth solution of

$$\begin{cases} u_t(x,t) - \Delta u(x,t) + c(x)u(x,t) = 0 & \text{in } U \times (0,\infty), \\ u = 0 & \text{on } \partial U \times [0,\infty), \\ u(x,0) = g(x), & \end{cases}$$

where U is a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂U.

a) If $g(x)$ is bounded and there exists a number γ such that $c(x) \geq \gamma > 0$ for all $x \in U$, prove
$$|u(x,t)| \leq Ce^{-\gamma t},$$
for some constant C.

b) If $g \geq 0$ and $c(x)$ is bounded, show that $u \geq 0$.

5. (20 points) Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where g, h are smooth functions with compact supports. Show that there exists a constant C such that

$$|u(x,t)| \le C/t,$$

for all $x \in \mathbb{R}^3$, $t > 0$.