臺灣大學數學系
九十七學年度下學期博士班資格考試題
科目：偏微分方程　　2009.02.26

1. (10 points) Solve
$$u_x + u_y + u = e^{x+2y},$$
with $u(x,0) = 0$.

2. (30 points) Assume $g \in C_c^2(\mathbb{R})$ and $h \in C_c^1(\mathbb{R})$.

   (1) Solve the initial value problem for the wave equation in one dimension:
$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g,\ u_t = h & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$

   (2) For the solution of the above equation, let $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ be the kinetic energy and $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$ be the potential energy. Prove

   (a) $k(t) + p(t)$ is constant in t,

   (b) $k(t) = p(t)$ for all large enough times t.

3. (20 points) Let $1 \le p < n$. If we want to establish an estimate of the form
$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|D^2u\|_{L^p(\mathbb{R}^n)},$$
for certain constants $C > 0$, $1 \le q < \infty$ and all functions $u(x) \in C_c^\infty(\mathbb{R}^n)$, what is the algebraic equation that p, q and n should satisfy? Give a quick proof to your answer by scaling the function u. The point is that the constants C and q should not depend on u

4. (20 points) Let
$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n,\ t > 0, \\ 0 & x \in \mathbb{R}^n, t < 0. \end{cases}$$
Assume that $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define
$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy.$$
Prove that

   (1) $\int_{\mathbb{R}^n} \Phi(x,t)dx = 1$,

   (2) $\lim_{(x,t)\to(x_0,0^+)} u(x,t) = g(x_0)$ for each point $x_0 \in \mathbb{R}^n$.

5.(20 points) Solve the initial-value problem for the quasilinear parabolic equation:
$$\begin{cases} w_t - \frac{1}{2}w_{xx} + \frac{1}{2}w_x^2 = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ w(x,t) = x^2 & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$