臺灣大學數學系
九十七學年度上學期博士班資格考試題
科目：偏微分方程　　　　2008.09.19

1. Let $u(x,t)$ satisfy

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, x > 0, t > 0$$

and the initial value condition $u(x,0) = 1$ for $0 < x < 1$, $u(x,0) = 0$ for $x \geq 1$.
(a) Find the entropy solution with the boundary data $u(0,t) = 1, t > 0$;
(b) Find the entropy solution with the boundary data $u(0,t) = 1/(1+t), t > 0$.

2. Solve the equation

$$u_{tt} - u_{xx} = 3e^x \text{ for } -\infty < x < \infty, -\infty < t < \infty$$
$$u(x,0) = 0, u_t(x,0) = 0.$$

3. Find a harmonic function $u(x,y)$ in the half-plane $\{y > 0\}$ with the boundary data $u(x,0) = 1$ for $x > 0$, $u(x,0) = -1$ for $x < 0$. (Hint: consider $u(x,y) = g(x/y)$.)

4. Let $u(x,t)$ satisfy

$$u_t = u_{xx}, 0 < x < 1, t > 0$$

with $u(0,t) = u(1,t) = 0$ for $t > 0$ and $u(x,0) = f(x)$ for $0 < x < 1$. Assume that $f(x)$ is continuous on $[0,1]$ and $f(0) = f(1) = 0$.
(a) Use the heat kernel and $f(x)$ to write down the solution of the equation.
(b) Show that

$$\lim_{t\to\infty} u(x,t) = 0.$$

(c) Show that there is $\alpha > 0$ such that

$$\int_0^1 u^2\,dx = O(e^{-\alpha t}) \text{ as } t \to \infty.$$

5. Let $\triangle u = 0$ in R^n.
(a) Let Du be the gradient of u and $v = |Du|^2$. Show that v is a subharmonic function ($\triangle v \geq 0$).
(b) Let $B_r = \{x : |x| < r\}$. Show that

$$\frac{d}{dr}\left(\frac{1}{r^m}\int_{B_r} |Du|^2\,dx\right) \geq 0 \text{ for } 0 \leq m \leq n.$$

(c) Show that

$$\frac{d}{dr}\left(\frac{1}{r^{n-2}}\int_{B_r} |Du|^2\,dx\right) = 2r\int_{\partial B_1} |\omega \cdot Du(r\omega)|^2\,d\omega.$$