國立臺灣大學數學系

九十六學年度上學期博士班資格考試題

科目：PDE

2007.09

1. Consider the first order PDE

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \ -\infty < x < \infty, \ t > 0.$$

   (a) (10%) Find the solution with initial data $u(x,0) = x$.

   (b) (10%) Find the entropy solution with initial data $u(x,0) = 1$ for $0 < x < 1$ and $u(x,0) = 0$ otherwise.

2. (20%) Consider the Laplace equation $\triangle u = 0$ in a two-dimensional fan Ω: $0 < r < 1$, $0 < \theta < \alpha$, where $0 < \alpha < \pi/2$ is the angle of the fan. Use separation of variable to find general solution for the Dirichlet problem

$$u(1,\theta) = g(\theta), \ 0 < \theta < \alpha,$$

   and $u(r,0) = g(0)$, $u(r,\alpha) = g(\alpha)$ for $0 \le r \le 1$. Here g is a smooth function.

3. Cosider a domain $\Omega = (a,b) \times (c,d)$ in two dimensions and an arbitrary C^1 function u defined on Ω with $u = 0$ on $\partial\Omega$.

   (a) (10%) Prove the Poincaré inequality: there exists a constant C which is independent of u such that

$$\int_\Omega |u(x)|^2 \, dx \le C \int_\Omega |\nabla u(x)|^2 \, dx.$$

   (b) (10%) What is the best constant C?

4. (15%) Consider the heat equation $u_t = u_{xx}$ on half line $x > 0$ and $t > 0$. with the boundary condition

$$u_x(0,t) = \alpha u(0,t), \ u_x(\infty,t) = 0, \ \text{for } t > 0.$$

   and initial condition $u(x,0) = f(x)$. Here, α is a constant and f is a smooth function with $f(\infty) = f_x(\infty) = 0$. Use heat kernel to construct solution.

5. (15%) Consider the telegraph equation:

$$u_{tt} + u_t = u_{xx}, \ -\infty < x < \infty,$$

   with initial data $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.
   Show that the solution is unique. (Hint: by energy method)

6. (10%) Consider the elliptic equation

$$\triangle u + c(x)u = 0, \ x \in \Omega \subset \mathbb{R}^d, \ \text{compact}.$$

   Under what condition does u satisfies maximum principle on Ω? Prove your statement.