國立臺灣大學數學系
九十五學年度博士班資格考試試題
科目：偏微分方程

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1. Solve
(a) $(x-y)u_x + u_y = u$, $u(x,0) = x$;
(b) $u_t + u^2 u_x = 0$ for $t \geq 0$, $u(x,0) = 1$ if $x < 1$, $u(x,0) = 0$ if $x \geq 1$, where u is a weak solution of the equation.
(c) $u_t + u^2 u_x = 0$ for $t \geq 0$, $u(x,0) = 0$ if $x < 0$, $u(x,0) = 1$ if $0 \leq x \leq 1$, $u(x,0) = 0$ if $x > 1$, where u is the weak solution satisfying the entropy condition.

2. Let $u(x,t)$ satisfy $u_{tt} = \triangle u$ in $R^3 \times R$ and

$$U(x,r,t) = \frac{1}{4\pi r^2} \int_{|y-x|=r} u(y,t)\, d\sigma(y)$$

be the average of u over $\partial B(x,r)$. Prove that
(a) $U_{rr} + \frac{2}{r} U_r = U_{tt}$;
(b) $\sum_{i=1}^{3} U_{x_i x_i} = U_{tt}$;
(c) $E(t) = \int_{R^3} \int_0^\infty [U_r^2 + \sum_{i=1}^3 U_{x_i}^2 + 2U_t^2] r^2\, dr\, dx$ is independent of t if $u(x,0)$ has compact support.

3. Let $g(x)$ be a continuous function, $0 < g(x) < 1$, and $u_N(x)$ be the solution of $\Delta u_N = 0$ in $\Omega_N = \{x : 1 \leq |x| \leq N\} \subset R^n, n > 2$, $u_N = g(x)$ for $|x| = 1$, $u_N = 0$ for $|x| = N$. (a) Show that $0 < u_N < 1$ and $u_N < u_{N+1}$ in Ω_N, and $u = \lim_{N\to\infty} u_N$ exists. (b)Show that u is a solution of $\Delta u = 0$ in $\{x : 1 \leq |x| < \infty\}$, $u = g(x)$ for $|x| = 1$, $u \to 0$ as $|x| \to \infty$. (c) Prove $a < u(x)|x|^{n-2} < b$ for some positive a and b. (Hint: Consider $v = 1$ and $v = \alpha|x|^{-n+2}$ and use the maximum principle.)

4. Suppose u is a C^2 solution of $u_t = \Delta u$ in $R^n \times [0,\infty)$, $u(x,0) = g(x)$, $|u(x)| \leq e^{|x|}$. Show that (a) $lim_{t\to\infty} u(x,t) = 0$ for each x if $g(x) = (1+|x|)^{-1}$; (b) $u(x,t)$ is nondecreasing in t if g is convex.