臺灣大學數學系111學年度第2學期博士班一般資格考試

科目：偏微分方程

2023. 02. 23

1. (25%) Consider the initial value problem

$$\begin{aligned} &\partial_t u + u\partial_x u = 0, \quad (x,t) \in \mathbb{R} \times (0,1), \\ &u(x,0) = x^3 - x, \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

(a.) Show that the initial value problem above has a classical solution. (b.) Evaluate $\partial_t u$ and $\partial_x u$ at the point $(1, \frac{1}{2})$. (c.) Find a characteristic such that $|\partial_x u|$ tends to infinity along it as t tends to 1 from below.

2. (25%) (a.) Let U be an open set in $\mathbb{R}^3$ and V be a connected compact subset of U. Show that there exists a constant $C > 0$ such that

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y), \forall x, y \in V$$

for all nonnegative harmonic function u in U. (b.) Show that there exists a constant $C > 0$ such that

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y), \forall x, y \in \mathbb{R}^3$$

for all nonnegative harmonic function u in $\mathbb{R}^3$. (c.) Show that if u is a nonnegative harmonic function in $\mathbb{R}^3$ then u is a constant function.

3. (25%) Let Ω be an open convex bounded smooth domain in $\mathbb{R}^3$. (a.) Prove that there exists a conmstant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C\|Du\|_{L^2(\Omega)}$$

for all $u \in H_0^1(\Omega)$. (b.) Prove that for each $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H_0^1(\Omega)$ of the boundary-value problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

4. (25%) Let u solve the following initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h, & \text{on } \mathbb{R} \times t = 0, \end{cases}$$

where g and h are smooth functions with compact supports. Prove (a.)

$$\int_{\mathbb{R}} |u_x(x,t)|^2 + |u_t(x,t)|^2 dx = \int_{\mathbb{R}} |u_x(x,0)|^2 + |u_t(x,0)|^2 dx$$

for all $t > 0$.

(b.)

$$\int_{\mathbb{R}} |u_x(x,t)|^2 dx = \int_{\mathbb{R}} |u_t(x,t)|^2 dx$$

for large enough t.