臺灣大學數學系
109 學年度下學期博士班資格考試題
科目：偏微分方程 2021.02.25

1.(a)(10%) Let $B = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ and $S_\pm = \{(0, 0, \cdots, 0, \pm 1)\}$. Argue the solvability of the boundary value problem:

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad B, \\ u = 0 \ \forall\, x \in \partial B \setminus S_\pm, \ \ u = 1 \ \text{at} \ x = S_\pm. \end{cases}$$

(b)(15%) Let $u(x)$ be a complex-valued harmonic function in an open domain Ω. Assume $x_0 \in \Omega$ such that $\sup_\Omega |u| = |u(x_0)|$. Show that u is a constant.
(c)(15%) Now let $\Omega = \mathbb{R}^n \setminus \overline{B}$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $\Delta u = 0$ in Ω. Furthermore, assume

$$\lim_{|x| \to \infty} u(x) = 0.$$

Show that

$$\sup_\Omega |u| = \max_{\partial\Omega} |u|.$$

2. Consider the scalar hyperbolic equation

$$\frac{\partial u}{\partial t} = a \frac{\partial u}{\partial x} \quad \text{in} \quad 0 \le x \le 1, \ \ t \ge 0,$$

where a is a positive constant. Let $u(x, 0) = f(x)$ and assume that $f(x) \in C^1$ in $[0, 1]$.
(a)(10%) Let $g(t) \in C^1$, $t \ge 0$, be any given function. Please impose the boundary value $u(x, t) = g(t)$ on *suitable* boundary such that $u(x, t)$ is solvable for all $0 \le x \le 1$ and $t \ge 0$.
(b)(20%) Find the compatibility conditions on $g(t)$ and $f(x)$ such that $u \in C^1$ in $0 \le x \le 1$ and $t \ge 0$.

3.(10%) Let Ω be an open bounded domain in $\mathbb{R}^n$ and $0 \in \Omega$. Show that $f(x) = |x|^{-\alpha}$ belongs to $W^{k,2}(\Omega)$ whenever $k + \alpha < \frac{n}{2}$, where k is a nonnegative integer.

4.(20%) Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$. Define

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x)dx.$$

Let $1 \le p < n$. For any $u \in W^{1,p}(\Omega)$, derive the Sobolev-Poincaré inequality

$$\|u - u_\Omega\|_{L^{np/(n-p)}(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)},$$

where C is independent of u. (Hint: use the contradiction argument based on the compact embedding theorem to derive the Poincaré inequality and then by the continuous embedding $W^{1,p}(\Omega) \subset L^{np/(n-p)}(\Omega)$ to conclude the result.)