臺灣大學數學系
109 學年度上學期博士班資格考試題
科目：偏微分方程 2020.09.18

In what follows, Ω denotes a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

1.(20%) Let the vector-valued function space $\mathcal{H} := \{u \in L^2(\Omega;\mathbb{R}^n) : \|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} < \infty\}$. Show that the trace map $T : \mathcal{H} \ni u \to u \cdot N|_{\partial\Omega}$ is a continuous map from $\mathcal{H}$ *onto* $H^{-1/2}(\partial\Omega)$, where N is the unit outer normal of $\partial\Omega$.

2.(20%) Assume that Γ is a C^1 hypersurface inside Ω. We write $\Omega = \Omega^+ \cup \Gamma \cup \Omega^-$, where $\Omega^+ \cap \Omega^- = \emptyset$ and both are open. Let $a^\pm(x) \in C^0(\overline{\Omega^\pm})$ be two positive scalar functions and $u^\pm(x) \in H^1(\Omega^\pm)$ satisfy $\nabla \cdot (a^\pm(x)\nabla u^\pm) = 0$ in $\Omega^\pm$. Find the conditions on $u^\pm$ such that if we define

$$u(x) = \begin{cases} u^-(x), & x \in \Omega^-, \\ u^+(x), & x \in \Omega^+, \end{cases}$$

then u is a $H^1(\Omega)$ solution of $\nabla \cdot (a\nabla u) = 0$ in Ω, where

$$a(x) = \begin{cases} a^-(x), & x \in \Omega^-, \\ a^+(x), & x \in \Omega^+. \end{cases}$$

3.

(1) (20%) Let f be a positive harmonic function in $\mathbb{R}^n$ with $n \geq 2$. Then $f \equiv constant$. (Hint: use the mean value property)

(2) (10%) The same result in (1) holds if f is a positive harmonic function in $\mathbb{R}^2 \setminus \{0\}$.

(3) (10%) Does (2) remain true if $\mathbb{R}^2 \setminus \{0\}$ is replaced by $\mathbb{R}^n \setminus \{0\}$ with $n \geq 3$?

4.(20%) Let u be the solution to

$$\begin{aligned} &u_t - \Delta u = 0 \quad \text{in} \quad (0,\infty) \times \Omega, \\ &\frac{\partial u}{\partial N} = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega, \\ &u = u_0 \quad \text{in} \quad \{0\} \times \partial\Omega, \end{aligned}$$

where u_0 is a smooth function. Show that

$$\lim_{t\to\infty} \int_\Omega |u(x,t) - \bar{u}_{0,\Omega}|^2 dx = 0$$

with

$$\bar{u}_{0,\Omega} = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx, \quad |\Omega| = \int_\Omega dx.$$

(Hint: the following Poincare's inequality is useful:

$$\int_\Omega |f(x) - \bar{f}_\Omega|^2 dx \leq C \int_\Omega |\nabla f|^2 dx.$$

Set $v(x,t) = u(x,t) - \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$ and proceed.)