臺灣大學數學系
107 學年度上學期博士班資格考試題
科目：偏微分方程　　2018.09.14

1. Let U be a bounded connected open set in $\mathbb{R}^3$. Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.
(a) (12%) Show that

$$u(x) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} u dS = \frac{3}{4\pi r^3}\int_{B(x,r)} u dx,$$

for each ball $B(x,r) \subset U$.
(b)(13%) Show the strong maximum principal, i.e., if there is $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u,$$

then, u is a constant in U.
2. Prove or disprove each of the following statements.
(a)(10%) If $u \in W^{1,1}((0,1))$, then $u \in L^\infty((0,1))$.
(b)(10%) If $u \in W^{1,2}((0,1)\times(0,1))$, then $u \in L^\infty((0,1)\times(0,1))$.
(c) (5%)If $u \in C^1((0,1)) \cap C([0,1])$ with $u(0) = u(1) = 0$, then there exists a positive constant C independent of u such that

$$\int_0^1 u^2(x)dx \le C\int_0^1 (\frac{du}{dx})^2 dx.$$

3. Suppose u is a classical solution to the following initial-boundary value problem for a viscous Burgers' equation:

$$\begin{cases} u_t + uu_x = u_{xx}, & (x,t) \in (0,1)\times(0,\infty), \\ u(0,t) = u(1.t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in [0,1]. \end{cases}$$

Here, $g(x)$ is a smooth function such that $g(0) = g(1) = 0$. Show that (a)(15%) $\int_0^1 u^2(x,t)dx$ is decreasing. (b)(10%) $\int_0^1 u^2(x,t)dx$ tends to 0 exponentially.
4. Assume U is a bounded connected smooth domain in $\mathbb{R}^3$. A function $u \in H^1(U)$ is a weak solution of Neumann's problem

$$\begin{cases} -\triangle u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{1}$$

if

$$\int_U Du \cdot Dv dx = \int_U fv dx$$

for all $v \in H^1(U)$. (a)(10%) Demonstrate that the above definition is reasonable by showing that a weak solution of (1) actually solves (1) provided it is smooth enough. (b)(15%) Suppose $f \in L^2(U)$. Prove (1) has a weak solution if and only if

$$\int_U f dx = 0.$$