臺灣大學數學系

106 學年度下學期博士班資格考試題

科目：偏微分方程 2018.03.02

博士班資格考 (PDE) Spring 2018

1. (30%) Assume (u,v) is a C^1 solution of the following system

$$\begin{cases} u_t + v^2 u_x = 0 & \text{in } \mathbb{R}\times(0,T), \\ v_t + u^2 v_x = 0 & \text{in } \mathbb{R}\times(0,T), \\ (u,v) = (g,h) & \text{on } \mathbb{R}\times\{t=0\}, \end{cases}$$

where T is a positive constant, g and h are nonzero smooth functions with compact support. Find the conservation law for this system of equations. (20 pts) Must solution (u,v) become trivial if $g \equiv h \equiv 0$? Find and justify your answer. (10pts)

2. (40%) Consider the following problem:

$$\text{Minimize}\left\{\int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), \int_\Omega u^2 = 1\right\}, \qquad (1)$$

where Ω is a bounded smooth domain in $\mathbb{R}^n, n \geq 2$.

(i) Find the Euler-Lagrange equation of (1)

(ii) Can the Lagrange multiplier of (1) become zero? Justify your answer.

(iii) Prove that the minimizer of (1) exists

(iv) Use the minimizer of (1) to find a nontrivial solution of

$U_{tt} = \Delta U$ for $x \in \Omega, t > 0$ with $U \equiv 0$ at $t = 0$.

3. (30%) Let

$$E_\sigma = \text{Minimize}\left\{\int_{\mathbb{R}^2} |\nabla u|^2 + \sigma|x|^2 u^2 + \frac{1}{2}u^4 : u \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} u^2 = 1\right\}$$

for $\sigma > 0$.

(i) Prove that the minimizer of E_σ exists for $\sigma > 0$.

(ii) Find the limit $\lim_{\sigma\to 0+} E_\sigma$ and justify your answer.

(iii) Let u_σ be the minimizer of E_σ. Suppose that $v_\sigma = v_\sigma(x,t)$

is a solution of $\begin{cases} v_t = \Delta v & \text{for } x \in \mathbb{R}^2, t > 0, \\ v = u_\sigma & \text{at } t = 0 \\ v(\cdot,t) \in H^1(\mathbb{R}^2) \end{cases}$ for $\sigma > 0$. Is there any $t > 0$ such that $v_\sigma(\cdot,t)$ is also a minimizer of E_σ?

Justify your answer.