臺灣大學數學系
106 學年度上學期博士班資格考試題
科目：偏微分方程 2017.09.15

1. (40 %) For each of the following statement, you have to prove it or disprove it.

(1) Suppose $U \subset \mathbb{R}^3$ is a bounded smooth open set. If $u \in W^{1,p}(U)$ with $p > 1$, then $u \in L^{\infty}_{loc}(U)$?

(2) Let $U = (0,2) \subset \mathbb{R}$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 < x < 2. \end{cases} \tag{1}$$

Do you think u has a weak derivative in $L^1_{loc}(U)$?

2. (40 %) Let $\eta(x) \in C_c^{\infty}(B_1(0))$ be a nonnegative function with

$$\int_{\mathbb{R}^N} \eta(x) dx = 1.$$

Set $\eta_\epsilon(x) = \epsilon^{-N}\eta(\frac{x}{\epsilon})$.

(1) Suppose $u(x) \in L^p(U)$, where $U \subset \mathbb{R}^N$ is open and bounded and $p > 1$. Show that $\|\eta \star u - u\|_{L^p(\mathbb{R}^N)} \to 0$ as $\epsilon \to 0^+$, where $\star$ is the convolution operator.

(2) Suppose $u(x) \in W^{k,p}(U)$, where $U \subset \mathbb{R}^N$ is open and bounded, $p > 1$ and k is a positive integer. Show that there exists a sequence of functions $u_j \in C^{\infty}(U) \cap W^{k,p}(U)$ such that

$$\lim_{j \to \infty} \|u_j - u\|_{W^{k,p}(U)} = 0.$$

3. (10 %) Let $\Omega = B_1(0) \backslash \{0\} \subset \mathbb{R}^3$. Find 6 different solutions for the following equation

$$\begin{cases} \Delta u(x) = 0, & \text{for } x \in \Omega, \\ u(x) = 3, & \text{for } |x| = 1. \end{cases} \tag{2}$$

4. (10 %) Solve the equation

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{for } x_2 > 0, \\ u(x_1, 0) = -e^{x_1^2}. \end{cases} \tag{3}$$